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INTRODUCTION

I. OUR POINT OF VIEW

Loose talk about reform in the teaching of elementary mathematics has spread confusion and shattered the confidence of many teachers. Uneasiness and fear have developed. This mood has damaging effects in the classroom.

Teachers have been told they are teaching horse-and-buggy mathematics in a space age. New and esoteric language has begun to flood the literature and to be introduced into discussions. Some innovators have pointed the finger of scorn at traditional subject matter, pronouncing it "bad mathematics."

Relax! The mathematics our pupils have learned in traditional classrooms is good, sound, useful mathematics. It is good enough to play a central role in any moon shot. Traditional content in elementary school arithmetic is sound. It is at the core of man's advance, including his past and future bursts into space.

Our real concern is with finding solutions to two interrelated problems. How can we help pupils (a) to learn *more* mathematics, and (b) to develop a *deep appreciation* for this increasingly important part of our culture? All proposals and discussions should be evaluated in terms of their contributions to the solution of these fundamental problems.

The traditional content taught to pupils in elementary schools is completely valid. Our elementary schools must teach pupils how to carry out the fundamental operations with whole numbers and fractions and how to use these skills in solving practical problems. But the truth of the matter is that pupils simply have not mastered the traditional content. Many have too weak a grasp on facts such as that $7+2=9$ and that $8\times9=72$. (Most of these pupils will drop mathematics as soon as they can, losing contact with a tool that is essential in coping with the problems of the times.) The lack of mastery cannot be remedied just by giving more drill. Pupils will be able to master more of the traditional content only if we give them freedom to *invent* ways of finding answers, if we take steps to make sure that they *understand* what they are doing, and if they have ample opportunity to *use* what they learn.

The job of teaching more mathematics will become easier to the extent that pupils develop an appreciation for the subject. This appreciation can develop only when mathematics is learned with understanding and through active participation—it will not take place when mathematics is learned by rote. We believe that even inspired teaching of isolated fragments of mathematics will fail to produce pupils who understand and can use what they have learned.

Mathematics is a world of endless fascination. It is a world brimming with surprises, patterns, and enticing vistas. To see a pattern emerge from a conglomeration of facts, to organize seemingly disconnected ideas—these

are some of the joys of learning mathematics. It is this sense of excitement in mathematics that MATH WORKSHOP wants pupils to experience in elementary school.

To accomplish this, MATH WORKSHOP puts its *major subject matter emphasis* on *basic principles* and *unifying ideas* which tie together the facts and manipulative procedures of arithmetic. This emphasis simplifies the subject, makes it understandable, insures recall, and increases interest. Our *major pedagogical emphasis* is on teaching for discovery. We present content to pupils so that they are challenged to think about relationships, to look for patterns and clues, to draw logical conclusions for themselves, and to invent their own ways of finding answers.

II. THE SUBJECT MATTER OF MATHEMATICS

A. The Spiral Curriculum Concept in MATH WORKSHOP

The *24th Yearbook* of the *National Council of Teachers of Mathematics* talks about "the basic mathematical understandings which should be continuously developed and extended throughout the entire mathematics curriculum, grades K-12 and beyond." The concept of the spiral curriculum has been used in designing MATH WORKSHOP. At each level of the MATH WORKSHOP the main ideas of mathematics are introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

However, pupils are not expected to master such ideas as, say, multiplication of fractions or addition of positive and negative numbers the first time they meet them. They encounter such ideas again and again, building always more depth of understanding.

The spiral curriculum rejects the "building block" approach to arithmetic that has been used in the past. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas with repeated reconsideration at later times. It calls for continual nurturing of important concepts year after year as understanding gradually develops.

In the early grades the hope for absolute mastery must be limited to the basic facts of arithmetic. Here, complete mastery can and must be achieved. But where big ideas are involved, mastery does not come quickly. It develops slowly and is based completely on understanding.

B. The Unifying Ideas or Strands in MATH WORKSHOP

MATH WORKSHOP emphasizes nine unifying ideas in mathematics:

1. *Structure*
2. *Sets*
3. *Number and counting*
4. *Numeration*

5. *Addition and subtraction*
6. *Multiplication and division*
7. *Functions and relations*
8. *Geometry*
9. *Measurement*

The coverage of these ideas on the pupil pages, in the Teacher's Guide, and in the practice sheets is summarized in the chart on pages 48-49. Each Teacher's Guide page lists the unifying ideas considered on the corresponding pupil page and contains many suggestions for extending these ideas beyond what is called for on the pupil page. Let us consider each of these unifying ideas.

1. Structure

Suppose that you came upon some sentences from a Martian arithmetic book:

1) | ☆ □ = □ ☆ |
2) ⊓ ☆ ◪ = ◪ ☆ ⊓
3) (⊠ ☆ ⌜) ☆ □ = ⊠ ☆ (⌜ ☆ □)
4) (□ ☆ ◪) ☆ | = □ ☆ (◪ ☆ |)
5) □ ☆ ⊠ = □
6) ◪ ☆ ⊠ = ◪

Somehow you have found out that the symbols:

|, □, ⊓, ◪, ⊠, ⌜

represent different cardinal numbers. You guess that the symbol ☆ is an operation sign, and you wonder which operation it stands for. Sentences 1 and 2 convince you that ☆ stands for neither subtraction nor division, for these sentences suggest that ☆ stands for an operation for which order does not affect the results. Sentences 3 and 4 also support this conviction because they suggest that grouping does not affect the results. However, there is nothing about sentences 1-4 which would help you choose between addition and multiplication as the operation denoted by ☆. Perhaps the other sentences will help.

Sentences 5 and 6 turn out to be of little help. In fact, both of them suggest that ☆ could denote addition, multiplication, subtraction, or division. Fortunately, sentences 1-4 rule out both subtraction and division. If ☆ stands for addition, then ⊠ denotes the number 0; if ☆ stands for multiplication, then ⊠ denotes the number 1.

It turns out that, based on what can be learned from sentences 1-6, the operation sign ☆ could stand for addition or it could stand for multiplication.

We can't decide from sentences 1-6 whether it is addition or multiplication that is denoted by ☆ because these operations have certain properties in common and these properties are reflected in sentences 1-6. The *additive*

structure of the number system is similar to its *multiplicative structure.* Both operations are commutative (sentences 1 and 2) and both are associative (sentences 3 and 4). Also, each operation has an identity element (sentences 5 and 6).

Whenever you provide exercises for pupils that bring any of these properties to the surface, you are paying attention to the structural properties of the number system. For example, exercises such as the ones which ask pupils to fill the frames to make the following into true sentences:

$$8 \times 7 = \bigcirc \times 8$$
$$7 + (3 + 5) = (7 + \triangle) + 5$$

are exercises which help pupils develop an awareness of commutativity and associativity. Another such exercise is one which asks pupils to complete the addition *Cross Number Puzzle:*

4	3	
5	1	

Notice that the cell in the lower right-hand corner gets filled with the same number whether the pupil adds the numbers in the rows and then adds the sums:

4	3	7
5	1	6
		13

or adds the numbers in the columns and then adds the sums:

4	3	
5	1	
9	4	13

This invariance of the total highlights the rearrangement properties of addition.

Of course, the advice that one should give pupils exercises which bring structural properties to the surface does not necessarily mean that pupils

need to be told the names for these properties. A pupil who has discovered these properties by working with displays of physical objects has a sound understanding of the properties. He thinks of them almost as laws of nature. A young pupil who tells you that $7 + 3$ is the same number as $3 + 7$ *because* "addition is commutative" has surely been coerced into using these words. He knows that $7+3=3+7$ because any display of objects which represents $7+3$ can be rearranged (without changing the total number of objects) into a display which represents $3 + 7$.

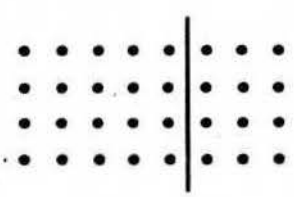
A pupil can be aware of and can use the structural properties without knowing their names. Exercises can be found which encourage pupils to make use of the properties. For example, a number line exercise which asks pupils to find the landing point when the starting point is 3 and the jump is 16 to the right could be solved by starting at 16 and jumping 3 to the right. The fact that this alternative procedure gives the correct answer strengthens a pupil's awareness of the commutative property of addition and helps to convince him of its usefulness.

If a pupil finds the sum of 16 and 3 by finding the sum of 6 and 3 and then finding the sum of 10 and 9, he has made use of the associative property of addition in a helpful way. (Of course, he has also made use of the fact that the numbers are expressed in decimal notation.) He does not become convinced of the correctness of his technique by being told that it is based on the fact that addition is an associative operation. Rather he becomes convinced by observing that this display of counters:

can be rearranged into this display:

In addition to the structural properties of commutativity, associativity, and the existence of an identity element for addition, the operation of adding a number has the property that it can be "undone" by subtracting the same number. Similarly, the operation of multiplying by a (nonzero) number can be "undone" by dividing by that number.

Pupils also learn that certain multiplication facts can be discovered by using other multiplication facts and addition. For example, pupils can find the product of 8 by 4 by adding the products of 5 by 4 and 3 by 4. That this procedure is valid is based on the "physical" fact that an 8 by 4 array of counters can be partitioned into a 5 by 4 array and a 3 by 4 array:



This experience helps pupils become aware of the fact that multiplication distributes over addition—a property of multiplication and addition. The term "distributive property" is not used on the pupil pages and need not be used in class. The important thing is that pupils use this property in deriving certain products from others.

2. Sets

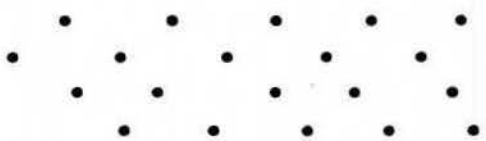
The principal contribution of sets to elementary arithmetic is that they can be used to clarify the notion of cardinal number. For a long time people thought that the word "two" in the expression "two apples" played the same kind of role as the word "green" in the expression "green apples." This was the cause of much confusion about the nature of cardinal numbers. An individual apple can have the property of "green-ness" but it can't have the property of "two-ness." Only sets can have the property of number. A set consisting of two green apples is a set whose cardinal number is two. Also, any set whose members can be matched with those of a set of two apples is a set whose cardinal number is two. From here it is an easy step to say that the cardinal number two is the class of all such sets.

Now, a set is an abstract entity just as a number is abstract. You can see two apples but you cannot see a set of two apples any more than you can see the number two or justice or liberty. For example, it is complete nonsense to ask the *set* of boys in the classroom to stand or to ask pupils to look at the *set* of counters on the table. There will come a time in the pupil's mathematical education when he will need to use the word "set" but we do not think it is a good idea for him to develop in the primary grades the notion that a set is a tangible object. Therefore, we have rarely used the word *set* in the Pupil Book, and have used it sparingly in the Teacher's Guide. Instead, we have used nontechnical terms such as *display*, *group*, and *array*.

Teachers who wish to read a discussion of set language and notation as used in mathematics can refer to the Appendix of this book as a supplementary unit of study.

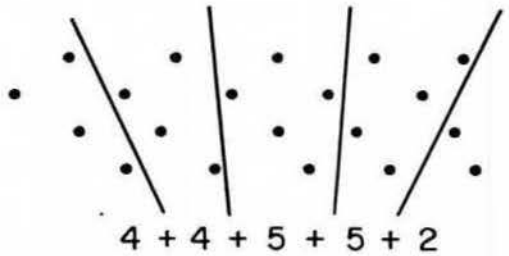
3. Number and Counting

In the primary grades the basic mathematical idea is that of counting. The operations of addition and multiplication are presented as shortcuts to counting. Given a display of objects:



and asked to tell how many objects are in the display, a pupil might proceed to count them one by one. This is the most primitive approach. A more

sophisticated approach is to partition the display, obtain the count for each of the smaller displays, and add the counts:



An even more sophisticated approach is to rearrange the objects in the display into a rectangular array:

If one is familiar with the count for a 4 by 5 array, he knows immediately that the original display contained 20 objects.

The study of fractions and negative numbers is considered an extension of the study of number and counting in this organization of unifying ideas.

4. Numeration

The idea of numeration grows out of the need to report a count, and the mode of reporting is culturally determined. For example, if the objects in the display shown above represent glasses of milk, we might report "1 gallon and 1 quart" in answer to the question: How many? If the objects represent eggs, we might report "1 dozen and 8 more" or "4 less than 2 dozen."

Each of these reports is a description of the result of some partitioning procedure. When decimal partitioning is used, the English language provides us with words for giving brief reports. For example, the report "3 ten-groups and 7 one-groups" or "3 tens and 7 ones" can be abbreviated to "thirty-seven" and can be recorded in positional notation by the symbol "37."

Much activity in the study of denominate (measurement) numbers is provided at all levels of MATH WORKSHOP to help pupils understand that the results of a count may be recorded in other bases than base ten. They come to realize that systems of numeration are arbitrary, but that each has a pattern which can be used to keep matters straight.

5. Addition and Subtraction

The operation of addition is developed as a shortcut to counting. Given enough opportunities to find the sum of 5 and 4 by counting, for example, a pupil soon commits to memory the fact that 5+4 is 9. He makes use of

this fact as well as of positional notation and associativity when finding the sum of 25 and 4:

$$25+4=(20+5)+4=20+(5+4)=20+9=29$$

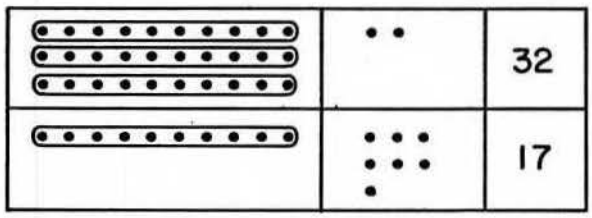
The addition facts from "0+0=0" through "9+9=18" should be memorized during the primary grades.

We use the word subtraction to denote the process used to undo the result of adding. So, for example, to compute 8−3 amounts to finding the number whose sum with 3 is 8. From this point of view, it is natural to learn subtraction facts at the same time one is learning addition facts. Thus, we have *fact teams*, groups of four sentences, such as:

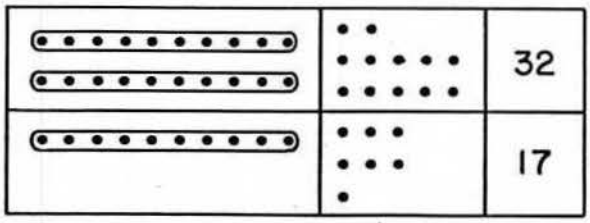
$$\begin{array}{ll} 5+3=8 & 8-3=5 \\ 3+5=8 & 8-5=3 \end{array}$$

which are derived from one addition fact with the help of commutativity and the fact that subtracting a number is the *inverse* of adding that number.

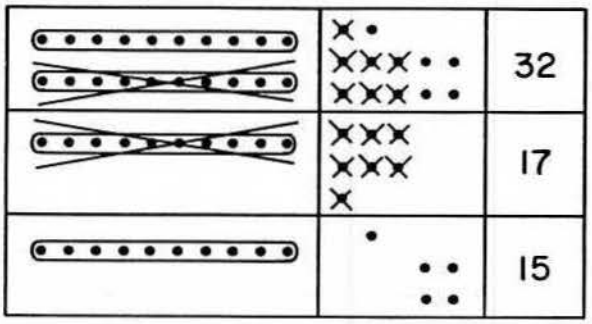
Addition and subtraction algorithms are developed as the end products of activities in which pupils manipulate physical objects and keep records of the manipulations. An algorithm becomes a record-keeping device for imagined manipulations. Thus, from the viewpoint of the pupil, an algorithm works because of laws which govern the manipulation of physical objects. For example, consider an algorithm used in subtracting 17 from 32. In terms of physical objects, one can find the difference by counting what remains when 17 objects are removed from a display of 32 objects. We start with decimally partitioned displays of 32 objects and 17 objects:



In order to carry out the take-away step, we need to break into one of the ten-groups. So, we rearrange the display for 32 by exchanging 1 ten-group for 10 one-groups:



and then take away 1 ten-group and 7 one-groups:

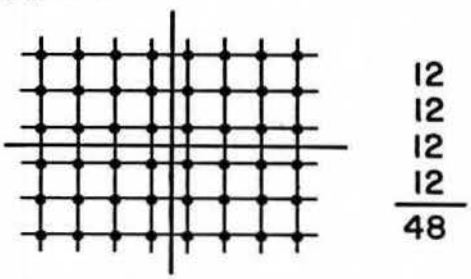


These three stages are recorded in the steps of the algorithm:

$$\begin{array}{rrr} 32 & 2\not{3}12 & 2\not{3}12 \\ \underline{-17} & \underline{-17} & \underline{-17} \\ & & 15 \end{array}$$

6. Multiplication and Division

Like addition, multiplication is developed as a shortcut to counting. In this case the counting involves objects arranged in rectangular arrays or otherwise grouped to encourage counting by groups. For example, counting the spaces in an egg carton and making 5 number line jumps of 2 to the right, starting at 0, are multiplicative situations. By continual practice in the primary years a pupil should commit to memory the multiplication facts from "$0\times0=0$" to "$9\times9=81$," and each of these facts should be discovered by the pupil. Some of the facts should be discovered by direct counting, but the discovery of others should involve partitioning of arrays and adding the counts of the parts. For example, a pupil might learn that 3×4 is 12 by counting the spaces in a certain type of egg carton. He can use this fact to learn that 6×8 is 48 by drawing 6 horizontal lines and 8 vertical lines and then partitioning the resulting display of crossing points:



Division is the process used to undo multiplication. So, for example, to compute $24\div8$ amounts to finding the number whose product with 8 is 24. From this point of view, it is natural to learn division facts at the same time one is learning multiplication facts. Thus, as in the case of addition and subtraction, we have facts teams for multiplication and division such as:

$$\begin{array}{ll} 8\times3=24 & 24\div3=8 \\ 3\times8=24 & 24\div8=3 \end{array}$$

which are derived from one multiplication fact with the help of commutativity and the fact that dividing by a (nonzero) number is the *inverse* of multiplying by that number.

7. Functions and Relations

An example of a function is the operation addition-of-2. This function assigns a unique number to each number. For example, it assigns 5 to 3, 19 to 17, and 202 to 200. A common way to exhibit a function is to show a table which lists some of the pairings. For example, to exhibit the function addition-of-2, we might show the table:

3	4	9	12	18	30
5	6	11	14	20	32

The numbers listed in the top row are the numbers on which the function operates. Those listed in the bottom row are the numbers we obtain from applying the function. The use of two colors helps to distinguish "input" from "output." By reversing input and output, this same table shows another function—subtraction-of-2—which is the inverse of addition-of-2.

All functions are examples of mathematical ideas known as relations. Numerical relations which are not functions are those which assign more than one number to some given number. For example, the greater-than relation assigns 4, 5, 6, etc., to the number 3.

8. Geometry

Under this general heading, one finds in MATH WORKSHOP a variety of topics—classification of common figures, comparisons of area and perimeter, partitioning of regions, similarity transformations (the doubling and halving of line segments), and the study of triangulation.

9. Measurement

Under this heading is found work with denominate numbers and the conversion of units. Further activity is provided in measurement of geometric shapes. There is a substantial amount of such work in Level D.

C. Another View of Mathematics: Counting and Shortcuts to Counting

You probably have noticed that a basic and simple notion recurs as a central theme in the previous discussion of the nine unifying ideas. It is that, to a large extent, elementary school mathematics is essentially *counting and shortcuts to counting.*

The authors of MATH WORKSHOP hope that all pupils will see arithmetic as a very simple subject. Pupils will develop basic confidence in their ability to move ahead if they are helped to realize that they have but a few easy tasks:

1. Learn how to COUNT
2. Learn how to RECORD RESULTS of counting
3. SOLVE PROBLEMS involving counting
4. Find STRATEGIES to avoid one-by-one counting

That's all there is to it!

Every pupil in elementary school ought to adopt the slogan: When in doubt, COUNT. He ought not look to the teacher, or the text, or his friends for assurance about his statements in arithmetic. Rather, he ought to look at the world of things . . . What happens there?

He ought to learn that there are many reliable ways to get to the solution to any exercise or problem—that he ought to use the way that seems most reasonable. You can help him understand this important freedom to find his own way and to use it. We try to give the pupil the equipment he needs to solve any problem so that, given enough time, he can, if all else fails, count his way to the answer.

Thus, our central purpose has been to give every pupil, from the fastest to the slowest, the confidence that he can always get right answers in computations. He must learn the strategies that are available to make the job easier, but he must understand them. It is the central purpose of MATH WORKSHOP to help pupils develop the confidence and ability to perform the four tasks enumerated above.

Lack of confidence is usually the real stumbling block to those who are called "slow learners." They are convinced they can't remember or they are afraid they will forget, so they try to avoid doing arithmetic. It never occurs to many of them that they can *always be right* if they count, and that they need use only those shortcuts or strategies they understand.

Slow learners will be given a new lease on their mathematical lives as we help them realize that there is less to memorize in this subject than any other. It all begins with counting and with thinking about what there is to be counted. Beyond that, if they learn the 390 basic facts of addition, subtraction, multiplication, and division, they can develop really effective strategies to avoid the hard work of one-by-one counting.

Fast learners need to learn about the other side of the coin. Since there is little that needs memorization and lots to think about, it is impossible to memorize your way through the subject. At some point along the way, the mind will rebel at all the memorization required—and the person, regardless of his ability to memorize, will become a "mathematical drop-out." The fast learner needs to realize that, since the subject is basically so simple, he can make important side-trips on his own, unafraid of pitfalls, confident he won't get lost.

III. HOW TO TEACH MATHEMATICS

A. Adopt a Sound Learning Theory

Every conscientious teacher has a theory of how pupils learn arithmetic. It is the sum of all the reasons behind his behavior in the classroom. If he asks the pupils to open the book to page 42 and do the even examples, he feels that activity will promote learning. If he tells Billy to write "$4 \times 7 = 28$" twenty-five times, he is convinced that the repetition will help Billy remember that fact the next time he is asked for the product of 4 and 7. If, instead, he asks Billy to count the number of days in 4 weeks, he feels that relating multiplication facts to recognizable events in the world he lives in will help Billy remember that $4 \times 7 = 28$.

If a teacher could get outside himself and analyze his teaching of just one mathematics class, he could then write down the theory of learning that guides him.

The authors of MATH WORKSHOP have a viewpoint on learning in mathematics. They recognize three different levels of learning experiences:

1. Rote Learning
2. Problem Solving
3. Independent Investigation

They believe that most failures and difficulties in arithmetic are the result of a conscientious, but misguided, emphasis by teachers and textbooks on rote memorization. The authors believe that the central task before us is to present arithmetic as a problem-solving activity and with unlimited opportunities for independent investigation.

1. Rote Learning

We share the ability to learn by rote memory with all animals. Pavlov's dogs learned by endless repetitions. Horses have been trained to point a hoof at 5 when the trainer calls out "two plus three." A psychologist, half in fun, pointed out that a stick also learns in this way—put a notch in a stick today and it will be there tomorrow.

Pages of random repetitions hope to educate students on the level of rote memory. If one page doesn't do the job, then let's try two. If pupils persist in forgetting, drill some more!

Actually, when pupils rebel openly or in any of the less obvious ways they are adept at inventing, that rebellion is usually a sign of an active mind. The human mind, after all, has a sense of dignity that is offended by long stretches of drill bent on rote memory as the objective. We should not be surprised when pupils turn away from assignments that recognize only the lowest level on which the human mind can operate.

Efforts to teach by rote memory have always been discouraging. Pupils don't respond "properly," so they are given larger doses of the same. In

desperation, the driller sooner or later reaches the conclusion that the pupil can't learn. The situation is further darkened by the mistaken theory that slow learners can only learn on the level of rote memory.

2. Problem Solving

The human mind is unique in its ability to solve problems—to face a question and announce: I don't know, but I can find out!

First graders are protected from questions such as "19+3=____" because the rules one needs to solve this problem by rote memory are not yet available. However, if a first grader can count as far as 22 and if he understands what is meant by addition, while he doesn't know the answer, he can find it. A consequence of teaching arithmetic at the level of rote memory in first grade was revealed when one of the authors asked a large group of beginning second graders to answer this question, asked in both forms they might have encountered:

$$\begin{array}{r} 19 \\ +\ 3 \\ \hline \end{array} \quad \text{and} \quad 19+3=\_\_\_\_$$

None were familiar with the horizontal form. All wrote the same result . . . without exception:

$$\begin{array}{r} 19 \\ +\ 3 \\ \hline 112 \end{array}$$

They read this statement as: Nineteen plus three equals one hundred twelve. They were asked: If you had nineteen cookies and I gave you three more, would you have one hundred twelve cookies? Most answered: No! I would have twenty-two cookies.

These second graders had been deprived of the opportunity to solve problems and were already victims of their most unreliable faculty—the ability to learn by rote.

First graders can understand and solve problems designed to give them opportunities to use the simple facts of addition. Here is a question that almost every adult has to think about:

> How many ways do you know to complete the following as a true statement?
>
> $$\_\_\_\_+\_\_\_\_=3$$

(We assume the pupil's acquaintance with numbers is limited to zero and the counting numbers.)

The pupil does not know the answer to the question . . . but he can find out—unless he has been trained to limit his response to "I don't know" when

he doesn't remember. A pupil who has often been encouraged to find out when he can't remember will make a list. He may write:

$$2+1=3$$
$$0+3=3$$
$$1+2=3$$
$$3+0=3$$

Is that all there are?

In response to this question, one first grade pupil simply reorganized the list without saying a word and took her seat.

$$3+0=3$$
$$2+1=3$$
$$1+2=3$$
$$0+3=3$$

This pattern was certainly an eloquent argument that there are no more!

Here is another first grade question that almost no adult can answer from memory. Yet it is a question that every adult and more first graders can work out if they make the effort:

> How many ways are there to complete the following as a true statement?
>
> $\_\_\_+\_\_\_+\_\_\_=3$

(Again assuming the problem solver is limited to zero and the counting numbers.)

Notice that in "finding out" one uses or practices basic combinations in addition. Other similar problems could require use of or practice in all basic combinations. All the random exercises in memory that constitute "drill" have now been encountered incidentally along the way toward the solution of problems.

> How many numbers can be expressed by one of the following or as the sum of two or more of them—none to be used more than once in any sum?
>
> 4, 2, 16, 1, 8

Only those who are already familiar with this list have any idea of the surprises that lie ahead. We begin a list, expressing each number as one from this list or as the sum of some combination of numbers from the list:

$1=1$	$6=4+2$	$11=\_\_\_\_\_$
$2=2$	$7=4+2+1$	$12=\_\_\_\_\_$
$3=2+1$	$8=8$	$13=\_\_\_\_\_$
$4=4$	$9=8+1$	$14=\_\_\_\_\_$
$5=4+1$	$10=\_\_\_\_\_$	$15=\_\_\_\_\_$

As we extend the list, we find no gaps until we come to 32. But that is one more than the sum of the entire list.

Now, we go back to see how many can be expressed in more than one way (not counting changes in order). We write:

$7 = 4+2+1$ ($7 = 2+4+1$ is considered the same combination)

Is there another combination from our list we might have used? One way to indicate 22 is $22 = 16+4+2$. Is there another combination whose sum is 22?

We try many combinations—using or practicing addition—and eventually develop a suspicion that each can be expressed in only one way as one of the numbers listed or as a sum of the numbers listed.

The authors of MATH WORKSHOP know that there is no end to such interesting problems in mathematics—problems that can be solved by pupils who are encouraged to say: I don't know, but I can find out.

We have assumed the responsibility of building a Pupil Book and designing practice sheets that will help teachers and pupils approach the subject of mathematics as being basically concerned with solving problems. When the pupil recognizes his ability to solve problems, he responds with enthusiasm, and his power to recall basic facts and processes he has encountered flourishes.

3. Independent Investigation

Once the pupil has developed a confidence that he can find things out for himself, he may be so delighted with his accomplishments that he will look around for problems to solve, and may invent problems of his own. He will not be satisfied with the list 4, 2, 16, 1, 8, which he used to write all the numbers 1 through 31. He will want to expand the list so he can go beyond 31. What number or numbers should he add to the list? Or, he might like to try an entirely different list such as 1, 4, 7, 9, 18, which behaves similarly but with important differences.

When pupils begin posing problems for themselves and solving them, they will have a taste of intellectual excitement that will lead them on and on. The teacher who has helped a pupil to use his mind on this level deserves the thrill that comes to him.

The authors of MATH WORKSHOP are dedicated to helping elementary school teachers rescue the subject of mathematics from emphasis on rote memorization and present it as basically a problem solving activity. Further, we have tried to present many opportunities that will encourage some pupils to pose questions to themselves and to solve them.

B. Pose Challenging Questions

1. "What Can You Say?"

The Report of the Cambridge Conference on School Mathematics said:

> "We hope that many problems can be found (we know a few) that read, 'Here is a situation—think about it—what can you say?' "

The authors of MATH WORKSHOP share that hope and have built the pupil pages, the accompanying practice sheets, and notes for teachers around such problems. A mathematical problem is a situation worth thinking and talking about, a situation that suggests exploration and investigation. Consider this situation:

What Can You Say? Situation 1

Make a list of the square numbers up to 100 . . .
0×0, 1×1, 2×2, 3×3, . . ., 10×10
0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

Making the list is simply computation . . . it is not a problem. But this list provides situations we can think and talk about. For the numbers in this list:

Is there a pattern of even and odd numbers? (Even and odd numbers alternate.)

By which square number (other than 1) can you divide evenly each even square number? (4)

What happens if we divide the odd square numbers by 4? (All odd square numbers give a remainder of 1 when divided by 4.)

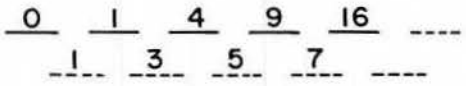
Can we find a pair of numbers in the list whose sum is also in the list? (Yes. Two such pairs are: $9+16=25$ and $36+64=100$.)

Are there any more? Could we find any more if the list were extended to include 121, 144, 169, etc.?

Can we find a number in the list that is twice as large as another in the list? Three times as large as another? (After a lot of work, providing meaningful practice, we find that the answer to both questions is "no!")

Is there a number in the list that is four times as large as another in the list? (Surprise!)

What can be observed about the differences between successive terms?



Will this pattern continue?

Each of the questions asked about this situation is simple and direct. If this is a pupil's first encounter with this situation, the answers have not been memorized. Yet each answer is well within the reach of any pupil who could have

created the list in the first place. He need only be ready to find out for himself by putting to work what he already knows.

How does one invent situations worth this kind of consideration? Unexpected sources for such situations are old arithmetic text books and current standardized tests . . . with a *little* modification. Both sources are, for different reasons, limited in their concern to simple, direct answers, so their questions do not lead to *situations worth considering.* They both suffer from a need or compulsion to give too much information. Here is an example: "Adult tickets are 85¢ each and children's are 45¢ each. *How much will 2 children's and 3 adult's tickets cost?*"

As is, there is nothing to think about . . . it only requires a bit of computation. The answer is $3.45 . . . and we go on to the next example.

This might be called the "customer" approach to such situations—one purchase, and that's the end of it. The clerk, or box office ticket seller, will have a different view of the matter. Here is his situation:

What Can You Say? Situation 2

Adult's tickets—85¢ each Children's tickets—45¢ each

He might make a pair of charts to help him solve the many problems he will face as different people make different purchases:

No. adult's tickets	1	2	3	4	5	6
Price	85¢	$1.70	$2.55	$3.40	$4.25	$5.10

No. children's tickets	1	2	3	4	5	6
Price	45¢	90¢	$1.35	$1.80	$2.25	$2.70

If his charts are long enough, he is now ready to compute each sale by looking up not more than two entries and adding them, perhaps in his head. If a customer wanted 3 adult and 2 children tickets, he would look at his charts

Adult's	3
	$2.55

Children's	2
	90¢

and ask the customer for $3.45. He might, however, make a chart to avoid even his addition. It would have to be quite a large chart. One corner of it would look like this:

Children 45¢ each

Adults 85¢ each	0	1	2	3	4	5
0	~~0~~	45¢	90¢	$1.35	$1.80	$2.25
1	85¢	$1.30	$1.75	$2.20	$2.65	$3.10
2	$1.70	$2.15	$2.60	$3.05	$3.50	$3.95
3	$2.55	$3.00	$3.45	$3.90	$4.35	$4.80
4	$3.40	$3.85	$4.30	$4.75	$5.20	$5.65
5						

One of his other charts is now in the top row of amounts. The other is in the left hand column. Other entries show the sums of the amounts at the top of the column and the left end of the row. If this chart were extended far enough, the ticket seller would have only to know the number of each kind of ticket to be purchased to find the entry giving the total amount to be charged for any purchase.

Notice that all entries in the chart above are different. So, if the amount of the purchase is known, one can find out exactly how many of each kind of ticket was purchased. If a purchase was \$2.20 and the chart was not available, it might require several trials to find that it could have been 3 children's and 1 adult's tickets and probably more trials to be sure that no other combination would fit.

We might ask if it is possible for a purchase to be exactly \$5.00. Without the chart, this would lead to a lot of computation before we found that the closest we can come to spending exactly \$5.00 is to spend \$4.95 for 11 children's tickets.

What is the smallest amount that can be spent in more than one way? (\$7.65 . . . 17 @ 45¢ or 9 @ 85¢.)

Remember that this investigation started from part of a typical "story problem." It was converted from a customer problem to an investigation of the box office salesman's point of view.

In MATH WORKSHOP Level D, a variation of this "clerk's approach" to developing situations is worked into the story of the *Arithmetic of Things and Numbers*. Ben and his sister are not satisfied to answer simple questions; they tackle a harder and more interesting problem than the one given—and include the answer to the simple question as an incidental part of their investigation.

At the bottom of many of the practice sheets, situations are presented that call for investigations; lines of attack are suggested, open-ended questions are asked, and work for the more capable pupils is laid out.

Here are several additional *What Can You Say?* situations, some taken out of mathematics itself, others taken from applications of mathematics to the real world. You can use each as a model for creating many others like it. Each suggests many similar situations that would bear independent investigation.

What Can You Say? Situation 3

Here are three paper cups:

Helen had four pieces of chalk. She put them in the cups.

Here is a situation. Think about it. What can you say? This and somewhat more pointed questions will elicit such observations as:

If two cups are empty, the other has 4 pieces in it.

There cannot be the same number of pieces of chalk in each cup.

There might be a different number of pieces of chalk in each cup—3 in one, 1 in another, and 0 in the third.

At least one cup has 2 or more pieces of chalk in it.

What can you say if Helen put only one piece of chalk in the cups?

Two cups are empty.

There cannot be the same number of pieces of chalk in each cup.

What can you say if she put 2 pieces in the cups?

There cannot be the same number of pieces of chalk in each cup.

They cannot all be different.

Two cups must have the same number of pieces in them—either two cups are empty or two cups have exactly one piece of chalk in each.

What can you say if Helen puts 3 pieces in the cups?

They can all contain the same number of pieces of chalk.

Each can have a different number of pieces of chalk —one empty, one with a single piece, and one with two pieces.

If two cups have the same number and the third has a different number, then two cups are empty.

What can you say if Helen puts 7 pieces of chalk in the cups?

They can't all contain the same number.

They can all have different numbers.

In at least one cup, there are at least 3 pieces of chalk.

If no cup has as many as 4 pieces in it, then one has 3 pieces and the others have 2 in each.

It could be that each cup that is to the right of another has twice as many pieces as the cup to its left. (1, 2, 4)

What Can You Say? Situation 4

Bill asked his father if he had change for a quarter. "Yes. What combination of coins do you want?"

Here is a situation. What can you say?

> How many of each kind of coin would Bill's father need in order to make change in any way Bill might ask? We might make a list of all possible combinations, following some plan that will help us know when we are done:

Dimes	2	2	1	1	1	1	0	0	0	0	0	0
Nickels	1	0	3	2	1	0	5	4	3	2	1	0
Pennies	0	5	0	5	10	15	0	5	10	15	20	25
Coins	(3	7	4	8	12	16	5	9	13	17	21	25)

> Bill's father would need 32 coins: 2 dimes, 5 nickels, and 25 pennies; with these 32 coins, he could make change in any of the above ways.
>
> There are 12 different ways to make change for a quarter.

(Do you agree with the authors that many of these explorations would make excellent independent activities, homework activities, or activities for exploration by teams of pupils?)

If we make a record (like the one above) of the number of coins in each combination, we notice that:

> Since no two combinations involve the same number of coins, we only have to know how many coins were used in order to tell which combination was used.
>
> There is no way to make change for 25¢ using combinations with exactly 2, 6, 10, 11, 14, 15, 18, 19, 20, 22, 23, or 24 coins.

How does the problem change if we investigate ways to make change for a half-dollar?

1. We already know that there would be 12 combinations using just one quarter (one quarter plus each of the 12 combinations listed above) and we know that there is only 1 combination with 2 quarters . . . a total of 13 thus far.
2. Without using quarters, we might list combinations using a specific number of dimes:

   5 dimes . . . . 1 combination
   4 dimes . . . . 3 combinations
   3 dimes . . . . 5 combinations
   2 dimes . . . . 7 combinations
   1 dime . . . . 9 combinations
   0 dimes . . . 11 combinations

3. This gives 1+12+36, or 49, combinations.
4. All amounts 30¢ and more can be made by at least 2 combinations that have the same number of coins.

30¢ . . . 5 pennies and 1 quarter (6 coins)
6 nickels (6 coins)
40¢ . . . 1 quarter and 3 nickels (4 coins)
4 dimes (4 coins)
1 quarter and 15 pennies (16 coins)
6 nickels and 10 pennies (16 coins)

Perhaps someone would like to investigate the ways to make change for $1.00 or $5.00. Or someone might notice that 90¢ can be made with at least 4 different combinations of 18 coins.

What Can You Say? Situation 5

SALE: Pencils . . . 5¢ each
Pens . . . 9¢ each
Notebooks . . . 25¢ each

Limit: 4 of each to a customer

What can you say about purchases that can be made under the terms of this sale?

What would be the largest purchase? (4 of each. Since 1 of each would be 39¢, 4 of each must be $1.56.)

Could a 30¢ purchase be made? (Yes. One way would be to buy a pencil and a notebook.) In what other way? (None)

How might the amounts indicated below have been spent? The record below shows one combination for each amount.

Amount of Purchase	23¢	53¢	76¢	59¢	81¢	89¢	$1.37
No. of Pencils	1	2	3	0	4	1	2
No. of Pens	2	2	4	1	4	1	3
No. of Notebooks	0	1	1	2	1	3	4

What other combinations of items, other than those indicated above, give purchases of 23¢, 53¢, 76¢, 59¢, 81¢, 89¢, or $1.37? (None)

Is it possible to spend $1.25 under the terms of this sale? (No) How many amounts of money can be spent in two or more different ways under the terms of this sale? (Surprisingly enough, every one of the

124 different combinations of items possible under the limit set has a different total price. Thus, if the amount of the purchase is known, the exact combination of items can be found.)

What amounts of money more than 10¢ and less than a dollar cannot be spent under the terms of this sale? (11¢, 12¢, 16¢, 17¢, 21¢, 22¢, 26¢, and 31¢. All other amounts up to $1.25 can be spent.)

Below is a summary of purchases of all combinations of pencils and pens, with no notebooks and with a single notebook:

No Notebooks

Pencils at 5¢

Pens at 9¢	0	1	2	3	4
0	~~0~~	5¢	10¢	15¢	20¢
1	9¢	14¢	19¢	24¢	29¢
2	18¢	23¢	28¢	33¢	(38¢)
3	27¢	32¢	(37¢)	42¢	47¢
4	(36¢)	41¢	46¢	51¢	56¢

One Notebook and

Pencils at 5¢

Pens at 9¢	0	1	2	3	4
0	25¢	30¢	35¢	(40¢)	45¢
1	34¢	(39¢)	44¢	49¢	54¢
2	43¢	48¢	53¢	58¢	63¢
3	52¢	57¢	62¢	67¢	72¢
4	61¢	66¢	71¢	76¢	81¢

Additional charts can be made, noting the combinations including 2 notebooks, 3 notebooks, and 4 notebooks. No amount will occur twice—all combinations of items have a unique total price; thus, if we know the amount spent and can find it in the chart, we can easily determine the items purchased.

Notice the five entries circled in the charts above: 36¢, 37¢, 38¢, 39¢, and 40¢—an over-two-and-up-one pattern (the Knight's move in chess). Also notice that 41¢, 42¢, 43¢, 44¢, and 45¢ follow the same over-two-and-up-one pattern. This may help suggest that no amount will be repeated.

If the prices are changed to

Pencils	3¢
Pens	4¢
Notebooks	16¢

and a limit set of 3 each to a customer, the items purchased by a given price are unique, as in the situation discussed above. The smallest purchase is 3¢. The largest purchase is 69¢. All amounts of money between 3¢ and 69¢ can be spent in one and exactly one way except 5¢, 64¢, 67¢, and 68¢—which cannot be spent for any combination of items.

What Can You Say? Situation 6

Suppose each of 6 pupils was asked to contribute his "favorite" number to a list and these numbers were recorded:

15, 4, 26, 1, 44, 9

What can you say about that list?

Three of them are odd and three are even.

None is prime.

Their sum is 99.

There is no combination of these numbers whose sum is 32, 33, or 34 (assuming that no number is included more than once in each combination).

$27 = 26 + 1$	$32 =$ _____
$28 = 15 + 9 + 4$	$33 =$ _____
$29 = 15 + 9 + 4 + 1$	$34 =$ _____
$30 = 26 + 4$	$35 = 26 + 9$
$31 = 26 + 4 + 1$	$36 = 26 + 9 + 1$

How many different numbers can be expressed as the sum of a combination of two or more of those listed? (57)

Can any member of the list be expressed as the sum of some combination of others from the list? (The answer is no. We can come close: $15 + 26 + 4 = 45$, but not 44; $1 + 9 + 15 = 25$, but not 26.)

Can any number be expressed as the sum of a combination of two or more numbers from the list in more than one way? (No. The sum of every combination is different from the sum of every other combination.)

Can you find another list of six numbers such that the answers to the last two questions are No? (The simplest is 1, 2, 4, 8, 16, 32.) There are an unlimited number of such lists but they are hard to find unless the numbers are separated by large gaps, for example: 1, 17, 101, 900, 2000, 7587. In fact, any list in which each new number is more than the sum of all preceding numbers will work. Here are a few that are not at all obvious:

4, 6, 7, 8, 20, 40
1, 4, 7, 9, 18, 36
1, 4, 6, 8, 16, 32

A newcomer to this problem will hurry through hundreds of examples of addition without noticing the arithmetic practice. His goal is the solution of a problem—the practice is incidental.

It's easy to ask a question so easy that almost anyone can answer it. And it's easy to dig up trick questions—there are books full of them. But challenging questions, like those in the *What Can You Say?* situations just presented, are those that encourage independent research and lead to discovery. They must be neither too hard nor too easy, and must not rely on

clever tricks. Rather, challenging questions are less than obvious but will yield answers if the pupil thoughtfully applies what he already knows.

Such questions are hard to come by. They are worth their weight in gold. Often they are hidden in very simple situations. In this section and throughout the MATH WORKSHOP materials, we have presented the largest collection of such problems that have ever appeared in elementary school textbooks.

2. *"What's My Rule?"*

In first grade, Miss Atkins tells the pupils that she has a secret rule (a function) and that she will help them read her mind. "Call out a number and I will answer with a number."

"Five," a pupil calls out.

"Four," Miss Atkins answers. "Someone call out another number."

"One," a pupil calls out.

"Zero," the teacher replies.

"Seven," another pupil calls out.

"Six," the teacher answers. "Have you guessed my rule?"

"Yes," Bill is quick to reply.

"Don't give my secret away. Just call out a number and tell me what you think I will reply."

"If I say 9, you are going to say 8."

"That's right. I *think* that you know my secret." Thus the pupils have learned a way to demonstrate their discovery.

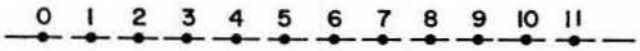
Many pupils are not as quick as Bill. So Miss Atkins may put on the board a record of the development thus far, first listing the number called out and then the reply:

$$5 \rightarrow 4$$
$$1 \rightarrow 0$$
$$7 \rightarrow 6$$
$$9 \rightarrow 8$$

The last item was Bill's contribution. Others might be:

$$6 \rightarrow 5$$
$$2 \rightarrow 1$$
$$4 \rightarrow 3$$

When the game is new or the rule difficult, a number line on the board will help:



"Let's act out the game we have played. When someone called out 5, I answered 4. Point to 5 on the number line. Keep your finger on that point. I answered by saying 4. Point to 4 on the line."

Each step is similary acted out.

"Now, if you call out 8, put your finger at the point on the number line."

$8 \rightarrow \_\_\_$

"What do you think I will reply?"

"Seven, of course!"

In some such way the idea can be made clear to even the slowest pupils. Care was taken not to talk about the teacher's rule itself—only about the results of applying it. Thus, every pupil must make the discovery for himself—some with minimum clues, others only after several different kinds of clues. There is a bit of artistry in giving clues only when needed and finding ways to reserve at least a little for even the slowest pupil to find out for himself.

Usually some pupil will note a curious fact about this rule.

"Suppose I call out zero. What will you answer?" a pupil may ask.

"Point to zero on the number line," the teacher suggests. "You can see that I would have to ask you to pick some other number." (Unless the teacher would like to talk about positive and negative numbers and zero.)

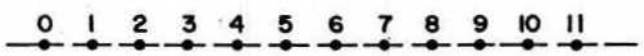
"I have a new rule. Call out a pair of numbers and I will reply by saying a number." Here is a record of a simple rule:

$2, 3 \rightarrow 5$
$4, 0 \rightarrow 4$
$3, 1 \rightarrow 4$
$1, 2 \rightarrow 3$
$1, 1 \rightarrow 2$
$3, 1 \rightarrow \_$

A more difficult rule is behind the following record:

$2, 4 \rightarrow 3$
$0, 2 \rightarrow 1$
$4, 0 \rightarrow 2$
$5, 9 \rightarrow 7$

This last rule may puzzle the pupils. Perhaps a number line can help shed light on the rule:



"If someone called out 2 and 4, I would reply with 3. Point to the numbers called out (2 and 4). I would reply with 3.

"If someone called out 4 and 0, I would reply with 2. Point to 4 and 0 on the number line. I would call out 2.

"Point to 5 and 9—and remember that I would answer 7. What would I answer if someone called out 1 and 7?"

$1, 7 \rightarrow \_\_\_$

"Four," Bill might answer, and quickly ask: "But what would you do if I called out 5 and 6 or if I called out 3 and 0?"

"I would have to ask you to pick another pair of numbers." (Unless the teacher is ready to discuss fractions.)

An unlimited number of rules can be devised on any level of sophistication. Further, there are many formats to be used in presenting rules, such as:

a. 1, 4, 7, 10, 13, , , . . .

b.

3	1	4	10	0	9	6
7	3	9	21	1		

c.

7	4	3	1
9	9	0	9
11	6	5	1
	8	6	
6			2

Mathematics is essentially a study of patterns. It has its own fascinating patterns and it is often employed to help reveal patterns in the physical world. When a scientist or any observer is searching for a better understanding of what he notices, he usually tries to set down a description of his observations. Whenever possible he will try to write down numbers that in some way help him discover relationships. He might record the following observations:

	1st	2nd	3rd	4th	5th
minutes	1	3	4	5	10
miles	1	9	16		

In his first three observations he notes time and distance. A pattern is strongly suggested. He may check his guess by making further checks for 5 minutes and 10 minutes. If his results are 25 miles and 100 miles, respectively, he will feel that he can formulate a rule or law that will suggest a pattern in certain real events ($1 \times 1 = 1$, $3 \times 3 = 9$, $4 \times 4 = 16$ . . .).

In mathematics we study patterns because they suggest basic principles of the subject itself and because mathematics helps record and explore patterns of the world we live in. The discovery of *one* basic principle can remove the necessity for learning many separate facts. Learning to discover patterns is a big step toward self-education. The recognition of the mathematical relationship among quantities in a scientific experiment allows further costly experimentation to be replaced by inexpensive computation.

Thus, this exercise or game of *What's My Rule?* has three important dimensions: (1) it can be used to motivate drill in the basic combinations of arithmetic; (2) it poses problems that pupils can think through to a solution; (3) it uses a method fundamental to the experimental sciences.

C. Provide for Individual Differences

As teachers, we strive to reach every pupil. We attempt to *intensify* the instructional process so that learning for all pupils, from the slowest to the

fastest, can proceed at an optimum level. Our strategy for this intensification begins with our concern for each pupil's *level*, *rate*, and *interest*. We further our aim as we choose activities and materials which, by their design, provide for the great variation of ability we find in our classes.

MATH WORKSHOP shares this goal with you—the intensification of the learning process for every pupil. In our concern about *level*, we have built *multi-level* materials into the program. To provide for differences in the learning rates of pupils, we have pioneered in developing problems in categories like *What Can You Say?* and *What's My Rule?*, which, by their nature, make it possible for the slowest pupil to progress while the most able pupil is inspired to investigate the questions more deeply. Our tests and suggestions for informal appraisal will aid you in making provision for the various levels and rates of your pupils. Further, we have developed problems and activities which may excite some pupils so much that they will ask whether they have to go to recess, or lunch, or some other coveted activity.

We have built MATH WORKSHOP with these ideas in mind. We have tested them on pupils for more than ten years. This present edition represents a third revision of the original material. The form of the present books, then, is a result of field testing with tens of thousands of pupils by the authors, their colleagues, and thousands of teachers from all parts of the country and in every conceivable type of school.

There are five specific ways in which we developed and organized MATH WORKSHOP so that its materials and suggestions would contribute to our common goal:

1. Through the Multilevel Structure of the Seven Basic Pupil Books

These Pupil Books were given the following designations: Beginner's Book, Level A, Level B, Level C, Level D, Level E, and Level F—so that at any grade a Pupil Book appropriate to the levels and rates of the pupils could be used. In general, the Beginner's Book would be used for Kindergarten or first grade, Level A for first grade, Level B for second grade, etc. However, a slow class or slow group in grade 3 might use Level A or B rather than Level C. Further, a very bright group may go on to Level D. Suggestions are made in the early pages of the Teacher's Guide to help you make this decision.

2. Through the Multilevel Structure of the Pupil Pages

Pages in the Pupil Book are designed with major consideration given to posing challenging questions. In the previous section we were able to demonstrate through examples how this approach to teaching provides for success for even the slowest pupil. At the same time, such a multi-dimensional technique opens the door to higher level responses and further investigations by the more able pupil.

3. Through the Multilevel Suggestions in the Teacher's Guide

Many suggestions are made throughout each Teacher's Guide for adjustments and extensions of activities to accommodate the various levels of achievement. Further, we should like to describe at this point four of the general ways in which individual pupil pages can be taught:

a. Use the page to conclude a development.
b. Use the page as a basis for discussion.
c. Let the pupils work in teams.
d. Let pupils work independently.

a. In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching—with liberal use of manipulative materials, the chalkboard, and class discussion. The pupil page is then used late in the development to tie the idea down or simply to record the progress that has been made toward deeper understanding.

There are limitless ways to initiate lively activities around important and useful concepts. MATH WORKSHOP provides many such activities for you to choose from.

After handling manipulative materials, playing number games, and other such preliminary work, pupils are ready to move through the pupil page in the final few minutes of the arithmetic period.

b. In contrast, some pupil pages might be gone through very carefully with an entire class. Such a page or sequence of pages offers many opportunities to dig deeply into "What's going on here?"

c. The team approach involves two or more pupils working on their own. It is very much like the fourth approach except for the rustle of activity as pupils speculate on theories as how to proceed. Teams may include pupils of many different levels of ability. This allows pupils to learn from one another. A contest based on the rate and accuracy of the team's effort is often a good motivational device.

d. A fourth approach might be called *Do-it-all-by-yourself*. Begin it by saying: Open your book to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a "quiet day."

Often, the first question a pupil faces when he begins work on a pupil page is: What am I supposed to do? There are almost always sufficient clues provided so that with a bit of study pupils can discover the intent of the exercises. Traditional materials have provided too little incentive for pupils to figure out for themselves the nature of the business at hand.

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several pupils are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity under way for everyone. Let

time run out on the session if you wish, suggesting that those who haven't finished may wish to take the work home with them to finish.

4. Through the Tests and Suggestions for Informal Appraisal in the Pupil Book and Teacher's Guide

There are 12 pages of tests in Level D that will aid you in identifying the levels of pupils and that will suggest where further stress will be needed in teaching. The Teacher's Guide offers many suggestions for interpreting these tests.

5. Through an Index to Remedial Activities in Earlier Levels of MATH WORKSHOP

An index is provided in this guide indicating the pages in Levels A, B, and C that may be used for remedial work.

6. Through Supplementary Materials Developed by the Authors of MATH WORKSHOP

Described in Section IV below are a number of MATH WORKSHOP supplementary multilevel teaching aids which are available from Encyclopaedia Britannica Press. These include: (a) a teacher training text, *Discovery in Elementary School Mathematics;* (b) Practice Sheets; (c) MATH WORKSHOP Supplementary Activities; (d) Work Forms; (e) Mastery Tests of the 390 Basic Facts of Arithmetic; and (f) Games and Enrichment Activities.

Nowhere today has one program collected, invented, and brought together into a comprehensive form such a great variety of material aimed at the intensification of the learning process.

D. Use the Discovery Method

As we have noted repeatedly, each page in MATH WORKSHOP, the activities suggested in the Teacher's Guide, the practice sheets, and other aids are designed so that there are several levels on which pupils can participate—including opportunities for discovery and success for even the slowest learners.

The *What Can You Say?* type of activities we present are truly open-ended discovery exercises with challenge for even the most advanced and able pupils in the class. While some questions can be easily settled, more will have been left open. Some pupils will wish to explore these open questions on their own. Such pupils ought to learn to work independently, sparing the teacher's time for those who need more of it. Actually, what the more able pupils need most is encouragement and the opportunity to develop a taste for independent study. The highest level of learning has been reached when a pupil begins to pose problems of his own and to work toward their solution.

In MATH WORKSHOP there is extensive provision for introducing or practicing the basic facts of arithmetic and for solving problems. In most cases such practice is organized so that it leads to the emergence of a pattern or the solution of a problem.

Discovering such patterns is one of the most rewarding experiences in mathematics. It is called the "Aha!" or the "Gee-Whiz!" experience—a gasp, the light dawns! "I see it!" These are events for which teachers work long and hard. One of the most disheartening facts teachers must face is that some pupils simply "don't see it." Yet this is also a challenge—it calls for exercising skill in teaching. Slow learners need more clues—more obvious clues; but the finest teacher will reserve a wee bit for each pupil to "see" for himself. If the teacher's best efforts fail, he may have to give away the secret and try again next time. This is an art that does not come easily, but the eventual results are deeply rewarding.

When the results of work in arithmetic exhibit an unusual regularity, the inquiring mind looks for an explanation. One problem may be solved but others are suggested. This is an exciting characteristic of mathematics.

When pupils discover a relationship, they are excited and eager to stay with that discovery. They will readily put this new idea to work, showing more examples of *it*. They come alive, often with an "Oo-o-o! I've got it," or with a burst of industry using a newly-discovered relationship to reel off answers with speed and confidence.

What is the *it* they have glimpsed? Often, if we distract their attention with a request for them to *tell about it* or *put it in words*, we have selected a method that could not be better calculated to turn pupils against discovery.

That a pupil can move ahead by putting *it* to work successfully is certainly all the evidence we need in order to know that he *has it*. Our only justification for veering off in the direction of a word-pulling exercise is a feeling that *it* is a generalization that ought to be verbalized in order to be usable. But research does not support this. Transfer to future experiences is not necessarily increased by attempting to *verbalize a generalization*. Subvocal or nonverbal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, is our main goal.

When pupils experience an insight—when they leap ahead because they have discovered a pattern—we ought to be very careful about letting words into the act too soon. The pupils probably do not have the language needed to express their discovery in words. If they try, we must, of course, help them. But too often such help and explanation degenerate into hair-splitting, an activity far removed from the act of discovery.

Gertrude Hendrix of the University of Illinois Committee on School Mathematics, who has done much research on discovery in the teaching of mathematics, warns that imprecise verbalization may *mutilate* a discovery. On the other hand, when some pupils (and teachers) make discoveries, they wish others to know about it. But herein lies a danger. Whenever a person reveals his secret to one who has not been as fortunate, the latter is deprived of the delight of discovering it for himself.

In MATH WORKSHOP we invent ways in which pupils can share their discovery without revealing the secret. We have them whisper their new idea in the teacher's ear; we encourage them to make up additional examples which will provide further clues for those who haven't found a pattern that underlies several examples. But we urge children to keep the secret—for a while at least.

Once in a while we find pupils who have not invented shortcuts for solving certain kinds of problems. For example, suppose that pupils use counting by ones in addition problems which involve fairly large sums. Hopefully, such pupils will become discouraged and begin to look for easier ways of determining the sum, such as counting by fives. If such easier ways are discovered, the pupils should be highly praised and urged to demonstrate these techniques to the entire class. If these examples are not sufficient to shake all pupils loose from counting by ones, then you must step in and urge the conservative ones to drop the slower technique. Please temper the discovery method with common sense.

E. Use Representational Materials

When mathematics loses contact with the world of things and with the events out of which it came, the subject becomes mysterious to most pupils. If they do not understand that in mathematics we are discussing the way things happen as we manipulate objects, pupils must resort to rote memory. MATH WORKSHOP activities stress the fact that arithmetic provides a means of recording or reporting what happens in the physical world.

Many kinds of manipulative or representational materials are suggested and discussed in this Teacher's Guide. These include various types of counting materials, geometric shapes, number lines, etc. Perhaps the most accessible source of representational materials is the pupil pages themselves. We believe that all of elementary mathematics can be abstracted out of experiences in the physical world. The very word *workshop* in the title of our program indicates our belief in the value of learning mathematics by working with physical objects. In some cases, the pupil pages are the workshop—pupils work with the two-dimensional pictorial representations of physical objects and apparatus and use the forms provided to keep records. Frequently the teacher will decide that the pupils need to work with three-dimensional equipment. In those cases, the teacher may provide equipment like that pictured on the pupil pages or like that suggested in the Teacher's Guide.

F. Teach for Mastery of Basic Facts and Algorithms

Mathematics provides a language to help report events that happen in the world of things. As we have noted, the core of elementary school mathematics is counting. Because this process becomes tedious when many things must be counted, shortcuts to counting have been invented and improved. Thus,

elementary school mathematics is essentially concerned with counting and with shortcuts to counting.

To some who have not studied the statements that precede this section, many MATH WORKSHOP activities appear to be highly motivated exercises to provide practice in the basic facts, combinations, and processes of arithmetic. In the vernacular, they wear the garb of "sugar-coated drill." Actually, they are valid educational devices, because they help establish that automatic recall of fundamentals without which pupils are mathematical stutterers.

Every teacher knows that many pupils are mathematical stutterers because they can't easily recall the basic facts, and stumble over the algorithms. They hesitate and flounder; the discussion bogs down; they lose the argument; they miss the feeling of movement toward a solution; they cannot enjoy mathematics.

This area of trouble is so universally known that it is surprising so many are only casually concerned! There is much hue and cry to bring new and unusual content into the elementary school curriculum—numeration using bases other than ten, sets, properties of numbers, and other concepts—yet there seems to be silence about such a crucial matter as helping pupils master the fundamental combinations. Without this mastery their progress toward understanding mathematical ideas is very painful.

While we have emphasized the scope of our program in terms of nine unifying ideas, *the basic facts and algorithms of arithmetic are taught and retaught throughout the program in all basic and supplementary materials.*

IV. MATH WORKSHOP TEACHING AIDS

A. Teacher's Guide

The introduction presents a point of view and a rationale for our approach to teaching elementary school mathematics. You will find this useful in interpreting many of the things we have included in the pupil pages. You will also find it useful in explaining the program to parents.

Each pupil page is reproduced in the Teacher's Guide in full size and in full color with answers printed in red.

Each pupil page is accompanied by a facing guide page that is divided into sections. Let's consider these sections:

1. *Tasks.* This section contains a very brief description of what the pupil will be doing as he works on the pupil page. We suggest that you study the pupil page in a pupil's book and work on it before you read our description and the answers we give.

2. *Purpose.* Here we describe what a pupil is intended to learn as he works on the pupil page or as he carries out some of the activities described in The Lesson. (Naturally, our description contains a conservative estimate of what

might happen. Mathematics is full of surprises and so are pupils.) If you are aware of our purpose, you will find it easier to help pupils achieve it.

3. *Unifying Ideas.* In this section we list those of the nine unifying ideas mentioned on pages 10-11 to which the exercises on the pupil page and recommendations in the notes on the page in the Teacher's Guide can be expected to contribute. Please regard the list as a minimum statement, and be prepared for pupils who may wish to engage in activities that go beyond what we have described for the page.

4. *The Lesson.* In this section we describe one or more of the ways we might build a lesson around the pupil page. No teacher could possibly find time to use all our suggestions in any school year. Choose the ones which best suit your style of teaching, and vary your choices from year to year.

B. Practice Sheets

On each of the first 96 pages of the Level D Teacher's Guide, a practice sheet is presented (in reduced size) and discussed in the notes when the purpose and use of the sheet are not self-evident. These pages may be copied or adapted by the teacher. (They are also available in full size from Encyclopaedia Britannica Press.)

Every practice sheet contains the beginning or continuation of the investigation of a problem, the exploration of a mathematical situation. Generally, the activity is a different approach to the development found on the corresponding pupil page, an extension of it, or a self-contained activity that provides extensive practice. No page is just random drill. The bottom section of most practice sheets provides story problems and *What Can You Say?* problems that will challenge some of the most able pupils in any fourth grade class. Do not expect all pupils who try these problems to get them right without help.

C. Math Workshop Supplementary Activities

A package of supplementary activity worksheets is available for every level, A through F, from Encyclopaedia Britannica Press. Each package includes 32 copies of 64 different activity pages. These can be used independently by pupils for seatwork and homework. Able pupils might be given supplementary activity materials from advanced levels. Slow pupils might work on these materials from lower levels. Each set of supplementary activities includes a Teacher's Manual with brief suggestions for each page and an answer key.

D. Work Forms

Many of the interesting and versatile page-forms developed in MATH WORKSHOP texts have been extracted and are available from Encyclopaedia Britannica Press as Multilith masters called *Work Forms*. In a few minutes,

a teacher can convert one of these forms with pen or pencil into a rich supplementary activity for the whole class or for a group within the class. Providing, as they do, significant independent learning activities for creative seatwork and meaningful homework, and taking into consideration, as they do, the wide range of individual abilities within any class, the *Work Forms* should prove very useful to busy teachers.

The *Supplementary Work Forms Manual* itself suggests many ways in which these forms can be used to help develop important ideas (and, at the same time, provide more drill on the fundamental facts). Teachers can create similar activities by adding whatever numbers they want in a matter of minutes.

Countless other uses for the *Work Forms* are described in the pupil's texts and the Teacher's Guides for the other levels of the MATH WORKSHOP. The ideas in the particular adaptations found there are easily applied to any level of development.

The *Work Forms* provide the teacher with a flexibility and versatility never before available in an organized and integrated program. While they are not essential to the use of the MATH WORKSHOP texts, they are an invaluable supplement, an effective means of meeting individual needs, and a help to every teacher in expressing his own individual approach to helping pupils understand arithmetic.

E. Mastery Tests of the 390 Basic Facts of Arithmetic

These tests are contained in the last section of the Multilith *Work Forms* described in D above, and are also available as a separate test booklet of 27 pages. They include a battery of six tests, each of which presents all the basic facts of addition, subtraction, multiplication, and division. An answer sheet is provided so that pupils can check their own or their neighbor's work. In addition, an individual record sheet enables each pupil to chart his own progress in terms of accuracy and time required to recall all the basic facts.

F. Discovery in Elementary School Mathematics

This is a professional development textbook that suggests to teachers many ideas for teaching mathematics at every grade level. This book is divided into two major sections—Multilevel Research Activities and Multilevel Worksheet Activities.

Multilevel Research Activities present 27 exploratory experiences which can be introduced on the chalkboard and with simple teacher-made manipulative materials at any grade level. Each activity starts simply and spirals upward with extensions and elaborations that will appeal to the more mature and sophisticated mind. Each activity focuses on mathematical structure in a context of discovery learning.

The Multilevel Worksheet Activities section contains 24 units, each of which explores some of the unifying ideas of mathematics. Each unit begins

with a brief introductory statement regarding mathematical structure and discovery. Each unit includes six activity worksheets, scaled in difficulty so that a teacher can present the ideas discussed in the unit to pupils at any level. These pages may be copied or modified by teachers for classroom use.

G. Games and Enrichment Activities

We have prepared a manual on the use of representational materials, such as Cuisenaire®* rods and Geo-boards. These materials can be used as the basis for games and other enrichment activities, many of which are mentioned or introduced in this Teacher's Guide. They contain many unusual applications of the concepts considered in the pupil pages.

H. Tests in the Pupil Book

Scattered throughout MATH WORKSHOP, Level D, are pages that we suggest you use as tests. These are pages 11, 12, 36, 37, 52, 53, 64, 65, 80, 81, 96, and 97.

I. Informal Arithmetic Inventory

The first ten practice sheets in Level D are designed to serve as an informal inventory. Teachers have used such informal tests in reading and spelling to provide each pupil with material at his instructional level. You might wish to do the same thing in arithmetic.

The psychological principle of success in learning is the basic notion behind such an inventory or placement test. We all know that the ideal learning environment provides for heaping measures of success and challenge, and eliminates or minimizes frustration. Unfortunately, no matter how carefully textbooks are designed—including MATH WORKSHOP, which provides as much help as possible for pupils at different levels of ability—some pupils are frustrated unless they are able to work in easier books in the series. It follows too that some few should work in harder books if they are to be sufficiently challenged.

You may wish to initiate this year's program by having pupils complete the first ten practice sheets first. In this case keep a record of their percentage of accuracy on each page for the problems you have assigned. You will then be able to determine whether Level D is the appropriate text for each pupil or if a lower or higher level in the series is called for.

You will notice that we are measuring more than computational skill in these ten pages. Consistent with our promise, we place computational problems most often in a context which requires more than automatic recall. You will be able to learn what each pupil does when confronted by a situation with which he is not familiar. In this way you will have a basis for knowing how pupils go about solving new problems.

*Cuisenaire—Registered trademark of Cuisenaire Company of America, Inc.

We offer the following as a guide to the placement of your pupils as you observe them working on these ten pages: (1) if a pupil scores 0-50% on the series, he might be placed in the Level B pupil book; (2) if a pupil scores 51-70%, he might be placed in the Level C pupil book; (3) if a pupil scores 71-100%, he can begin in the Level D pupil book.

As with any evaluation technique for young pupils, care must be taken not to put more faith in tests than they deserve. Your own judgment of a pupil based on your face-to-face work with him is probably more reliable than scores on tests.

This brings up the general problem of marking or grouping pupils as they work on all of the pupil pages. The pupil pages should be considered opportunities for investigation and exploration on the part of the pupil rather than as pages to mark for a grade. There are many pages on which pupils are not expected to get 100% on the first try. Nothing would be more damaging to their regard for mathematical inquiry than to be pummeled with red marks and low grades. We suggest that you *avoid* marking pages with a *minus* score. The number right or the percentage of accuracy is a more positive approach. By all means, keep a light touch rather than a heavy hand when checking and marking these pages.

V. PARENT EDUCATION AND INVOLVEMENT

Ill-advised and misleading publicity in newspapers and magazines has frightened parents into believing they can no longer help their children at home. The practice sheets may help considerably to soothe these parents. No new mathematical language is used on any of these pages. Most require extensive mental or scratch-paper computations in addition to the results finally recorded on the sheets. None of these computations is beyond the experience of most parents.

A common cause of misunderstanding between parents and children is concern for doing a computation the "right way." The child will often argue that the book's way or the teacher's way is the "right way"—and the parent will prefer his way. Of course, there is no one right way. There are only right answers or right results or valid conclusions. The way one chooses is his own business, a matter of personal taste—so long as it does not lead him to nonsense. MATH WORKSHOP suggests many approaches to problems and urges teachers to place a high premium on the discovery of many ways to find a right answer. We have argued elsewhere that if a child does not have at least one alternate method of finding a right answer, then he does not understand the method he is using.

Parents can do much to help children move ahead in mathematics. If they work together with their children toward solutions of problems, avoiding questions of right and wrong methods, mathematics can become a common interest, contributing to a more constructive home environment. Parents may even come to realize that the fact that they "can't even keep the checkbook

straight" does not really indicate a lack of mathematical talent or prevent them from participating with enjoyment in mathematical investigations. In fact, in these activities, parents are likely to discover that mathematics is a part of their culture that was denied them by an early contact with a crude facsimile of the subject.

Practice sheet activities provide opportunities for children and parents to work together and to think together about problems—a much more productive experience than completing assigned tasks in a prescribed way. Suppose a question is brought home: How many ways are there to make change for a quarter? Out come the coins from father's pocket or mother's purse or the piggy bank. Someone must keep a record. How can you be sure that you have all the combinations? What kind of record will help answer that last question?

"Let's try a dollar next!"

And later, in the car, "I'm thinking of a combination of 9 coins that add up to 29¢. What coins am I thinking about?"

Practice sheets were designed with "home consumption" in mind. They may help bridge the gap that misinformation and rigid assignments can create. Of course, the Supplementary Activities and *Work Forms* can also be used in the same way.

VI. PUNCTUATION AND ORDER OF OPERATIONS

In Level D pupils will sometimes face expressions such as "$8-2+3$," which appear to be ambiguous. Does this expression stand for 9 or for 3? That is, should one subtract 2 from 8 and then add 3 to the result, or should one add 3 to 2 and subtract the result from 8? One way to remove the ambiguity is to punctuate the expression with a pair of parentheses. Thus,

$$(8-2)+3=6+3=9$$

and

$$8-(2+3)=8-5=3.$$

A second way to remove the ambiguity is to establish a convention or agreement. The convention we follow in MATH WORKSHOP is that when an expression involves addition and subtraction, one should proceed from left to right in carrying out the indicated operations, unless parentheses are used to indicate a contrary order. Thus,

$$8-2+3=6+3=9$$

and

$$9+4-1=13-1=12$$

and

$$7-2-1-3=5-1-3=4-3=1.$$

When an expression involves multiplication and addition or subtraction, the convention we follow says that, unless parentheses indicate a contrary order,

one should perform the multiplications first, and then the additions or subtractions from left to right. Thus,

$$2\times3-1=6-1=5,$$
$$4+3\times2=4+6=10,$$
$$3\times5+7\times5=15+35=50,$$
$$\begin{aligned}4\times2+5\times3-6\times2&=8+15-12\\&=23-12\\&=11.\end{aligned}$$

VII. A FINAL WORD

Makers of I.Q. tests search for measures of the pupil's ability to think. They try to rule out questions that merely assess achievement in subject-matter areas. Therefore, they would not ask pupils to complete the following familiar examples from arithmetic. These questions would more likely appear in an achievement test.

$\begin{array}{r}3\\+7\\\hline\end{array}$ $\begin{array}{r}41\\-16\\\hline\end{array}$ $\begin{array}{r}346\\\times\ 27\\\hline\end{array}$ $5\overline{)135}$ $8\times3=$

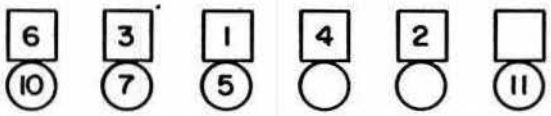
$\frac{1}{2}+\frac{2}{3}=$ $1.19=17\%\times\_\_$

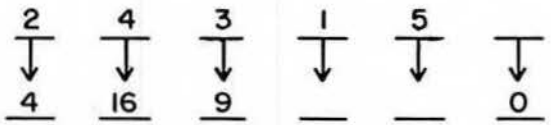
For measurement of I. Q. the questions from mathematics would be more like these:

I. Extend each of the following sequences:

a. 2, 5, 8, 11, 14, __, __, __ .....
b. 1, 2, 4, 8, 16, __, __, __ .....
c. 3, 4, 6, 9, 13, __, __, __ .....
d. 1, 1, 2, 3, 5, __, __, __ .....

II. Fill the blanks in the following examples:

a. 

b. 

c. 1, 3 → 8 ; 3, 2 → 10
5, 0 → 10 ; 6, 3 → __
2, 5 → 14 ; __, 4 → 12

III. Circle the number that seems most out of place in each row:

a. 6 , 10 , 2 , 7 , 8 , 4
b. 25 , 70 , 43 , 16 , 31 , 52
c. 13 , 15 , 19 , 3 , 11 , 7

Such problems, requiring a familiarity with arithmetic, are held to be more appropriate as measures of the ability to think. They are designed to test intelligence rather than achievement.

Mathematics is a fertile source of I.Q. test questions because mathematics is fundamentally a study of patterns and relationships. Many of the questions selected are similar to the examples given above. Behind that choice is the test-maker's estimate that pupils have not already explored and investigated such questions.

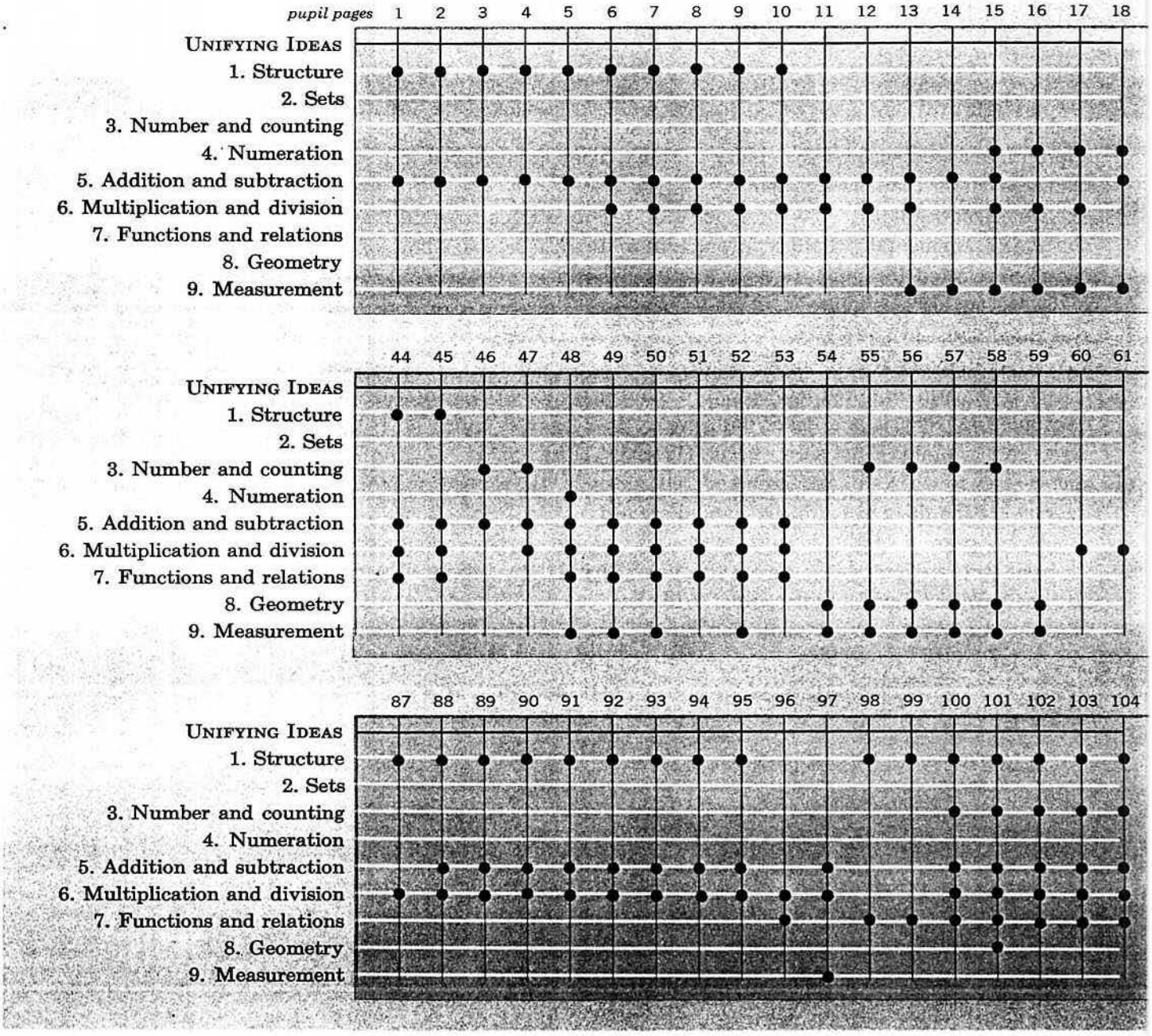
What an indictment of the elementary school curriculum in mathematics!

If arithmetic is limited to memory of number combinations and rules for manipulation, then it is a dead, dull, dreary task against which an active, eager, creative mind must surely rebel. However, if we broaden our thinking and look everywhere for patterns and relationships, the subject stimulates the mind to inquire and search out, and discover!

Every page in MATH WORKSHOP contains problems that probe beneath the basic facts and processes. We are always exploring the question: Why does arithmetic work? Each activity asks: "What can you say?" Each activity invites the pupil to ask: What's going on here?—and urges him to find out.

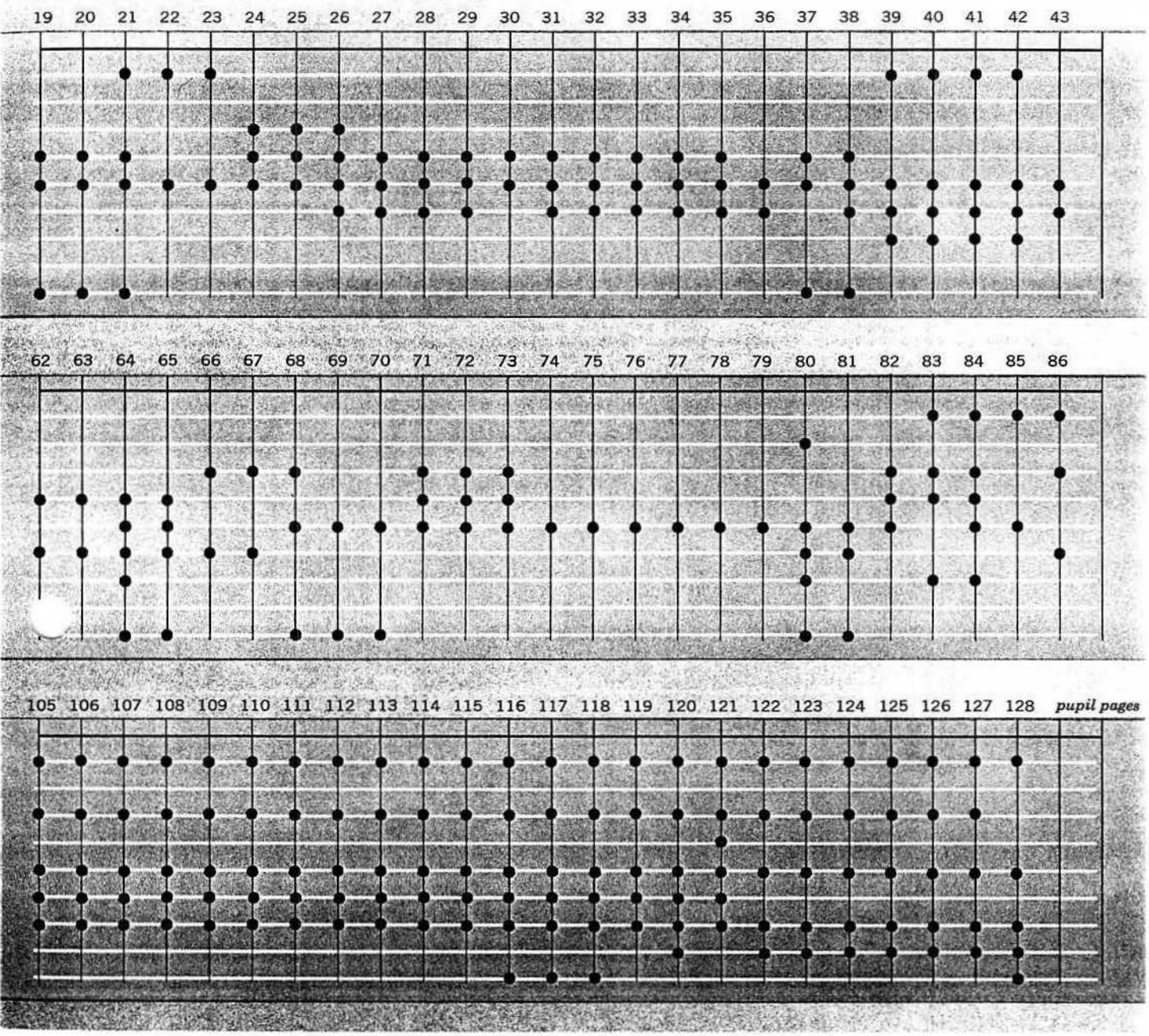
Robert W. Wirtz
Morton Botel
Max Beberman
W. W. Sawyer

Level D Revised


pupil pages 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
UNIFYING IDEAS
1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement
44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61
UNIFYING IDEAS
1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement
87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104
UNIFYING IDEAS
1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement

Consider each unifying idea . . . on how many pages will you find it?

1. Structure	63 pages
2. Sets	1 page
3. Number and Counting	47 pages
4. Numeration	33 pages
5. Addition and Subtraction	108 pages
6. Multiplication and Division	80 pages
7. Functions and Relations	48 pages
8. Geometry	15 pages
9. Measurement	33 pages



Consider each page . . . how many unifying ideas will you find on it?

8 pages contain just one unifying idea,
27 pages contain two unifying ideas,
45 pages contain three unifying ideas,
15 pages contain four unifying ideas,
27 pages contain five unifying ideas, and
6 pages contain six unifying ideas.

I Magic Squares: a Review of Basic Arithmetic Facts

Add the rows. Add the columns. Add the diagonals.

A

8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

B

10	12	2	24
0	8	16	24
14	4	6	24
24	24	24	24

C

2	11	5	18
9	6	3	18
7	1	10	18
18	18	18	18

When the row, column, and diagonal sums are the same, we say the square is MAGIC.

Fill in the green blocks so that each square of 9 green blocks is MAGIC. Show the row, column, and diagonal sums in the blue blocks.

D

1	8	3	12
6	4	2	12
5	0	7	12
12	12	12	12

E

5	5	2	12
1	4	7	12
6	3	3	12
12	12	12	12

F

2	1	3	6
3	2	1	6
1	3	2	6
6	6	6	6

G

0	2	1	3
2	1	0	3
1	0	2	3
3	3	3	3

H

3	0	3	6
2	2	2	6
1	4	1	6
6	6	6	6

I

11	17	2	30
1	10	19	30
18	3	9	30
30	30	30	30

Look for a connection between the number in the center block and the Magic total. In each row, column, and diagonal that includes the center block, what do you notice about the other two numbers in that row, column, or diagonal?

J

24	3	18	45
9	15	21	45
12	27	6	45
45	45	45	45

K

3	19	8	30
15	10	5	30
12	1	17	30
30	30	30	30

L

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Tasks: In exercises A-C pupils compute the row, column, and diagonal sums to determine whether or not the squares are Magic. In exercises D-L pupils use the clues provided to reconstruct Magic Squares. In exercises J-L pupils determine a relationship between the numbers in the center and the Magic sums.

Purpose: To perfect addition and subtraction skills for whole numbers by exploring several elementary properties of Magic Squares.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: Almost all pupils need more practice in computation. However, the setting for this practice is of tremendous importance. Rote drill is at best an unhappy chore that fosters a distaste for arithmetic and misrepresents it as essentially a challenge to the memory. A series of unrelated, pointless computations bores an active mind.

Practice sheets 1-10 form an *informal arithmetic inventory*. This inventory and criteria for pupil placement are discussed in the Introduction, pages 43-44.

Throughout MATH WORKSHOP pupils have many opportunities to practice computation in a setting in which it is a means to a larger end. Pupils' explorations of Magic Squares on the first ten pupil pages is a good case in point. Much computation is required—but attention is focused on more interesting activities, such as (1) testing an arrangement for the "Magic" property; (2) finishing an incomplete square; and (3) noting that Magic Squares can be produced by adding or subtracting Magic Squares or by multiplying or dividing Magic Squares by a number.

On pupil page 1 pupils are introduced to three arrangements with the "Magic" property—examples A, B, and C. In the other examples they must look for clues in forming Magic Squares.

They are encouraged to notice the relationship between the number in the center block of the square and the Magic sum. The sum is always 3 times the number in the center.

Here we consider only 3 by 3 Magic Squares—said to be of order 3. There are 4 by 4 Magic Squares, 5 by 5 Magic Squares, etc., but in the pupil pages and practice sheets we limit our investigation to those of order 3.

Practice sheet 1 introduces a variation of the pupil page activity—arrangements that are not Magic, arrangements with flaws. Such examples force pupils to check all rows, columns, and diagonals before deciding that an arrangement is truly Magic.

Your most able pupils can try completing Magic Squares in which only a few clues are given. Here are some that can be completed as Magic Squares in only one way:

1.			2.			3.			4.		
6	7	2	7	5	15	1	6	2	10	1	7
1	5	9	17	9	1	4	3	2	3	6	9
8	3	4	3	13	11	4	0	5	5	11	2

5.			6.			7.			8.		
18	21	6	25	0	35	8	7	3	7	8	15
3	15	27	30	20	10	1	6	11	18	10	2
24	9	12	5	40	15	9	5	4	5	12	13

9.			10.			11.			12.		
8	1	6	8	1	6	8	1	6	8	1	6
3	5	7	3	5	7	3	5	7	3	5	7
4	9	2	4	9	2	4	9	2	4	9	2

PS-1

Are they Magic? Check (✓) evidence of flaws.

			45				39✓						51		
12	27	6	45	22	2	16	40	29	1	21	51				
9	15	21	45	8	13	19	40	9	17	27	53✓				
24	3	18	45	10	25	5	40	13	33	5	51				
45	45	45	45	40	40	40	40	51	51	53✓	51				
			42✓				36				77				
23	0	18	41	15	14	7	36	31	5	41	77				
8	14	19	41	4	12	20	36	36	26	15	77				
10	26	4	40✓	17	10	9	36	10	46	21	77				
41	40✓	41	41	36	36	36	36	77	77	77	78✓				

Complete each as a "Magic Square" without flaws.

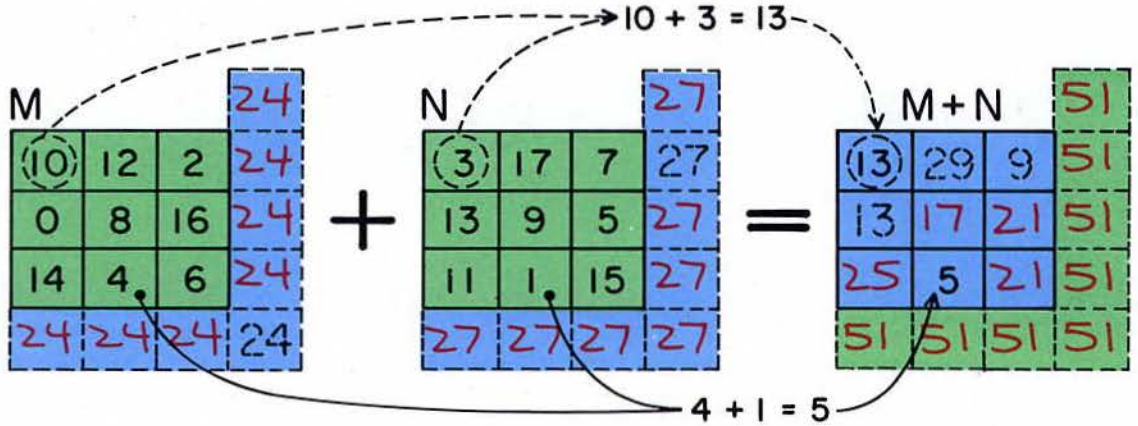
			15				30				45
8	1	6	15	16	2	12	30	24	3	18	45
3	5	7	15	6	10	14	30	9	15	21	45
4	9	2	15	8	18	4	30	12	27	6	45
15	15	15	15	30	30	30	30	45	45	45	45

Special on cupcakes 4¢ each 3 for 10¢

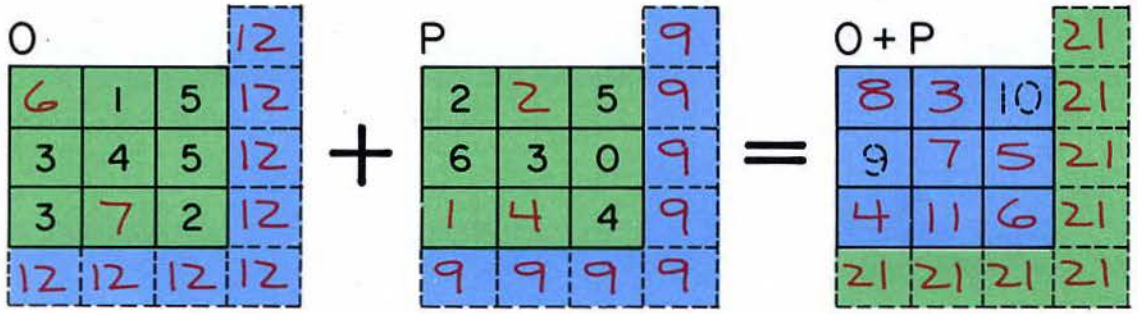
Two cupcakes would cost 8¢.
Mary wanted 4 cupcakes.
The clerk charged her 15¢, and gave her 11¢ change from her quarter.

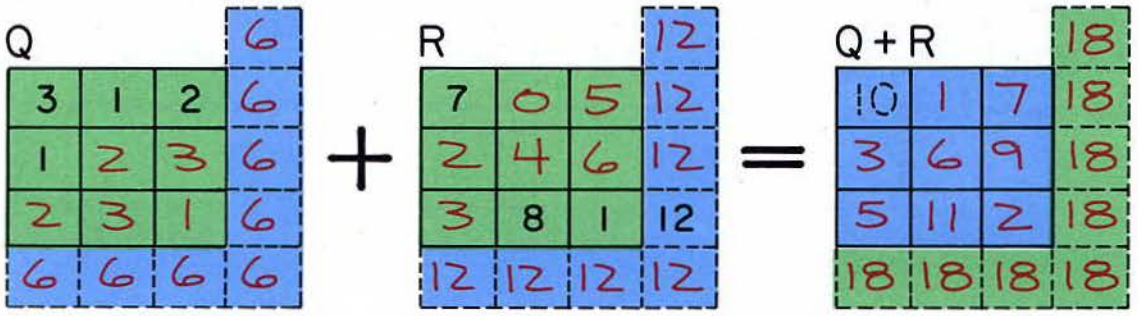
Mrs. Martin bought a dozen cupcakes for a party. She paid 40¢ for the dozen cupcakes.
Two dozen cupcakes would cost 80¢. A dozen and a half would cost 60¢, leaving 40¢ change from a $1 bill.

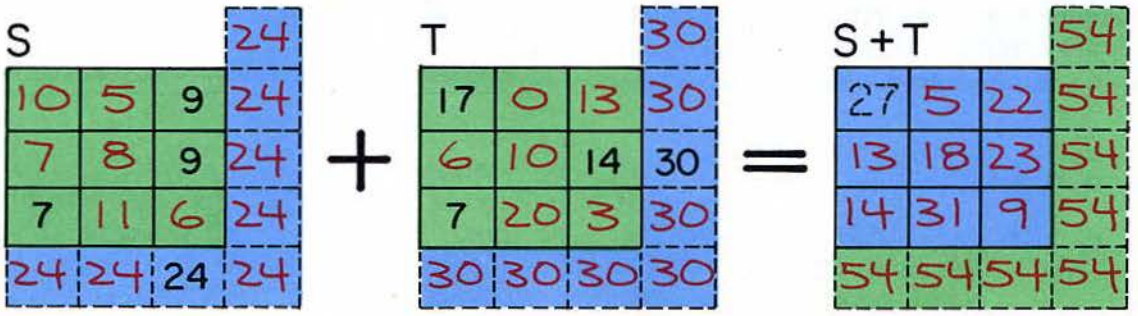
Harry told the class, "I've found an easy way to make a new Magic Square out of two old ones. I just add the old ones — that is, I just . . . well, here's what I mean!"



Let's try Harry's idea on other Magic Squares. Complete each of the first two (tinted green) as a Magic Square. Then "add" them as Harry did. Is the result also a Magic Square? You may want to try others on your own.







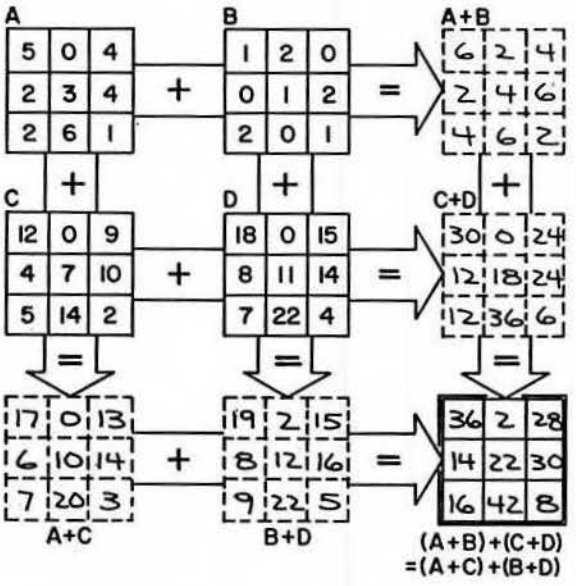
Tasks: *Pupils apply Harry's definition of addition for Magic Squares and use the clues provided to construct the Magic Squares O through T, (O+P), (Q+R), and (S+T).*

Purpose: To perfect addition and subtraction skills for whole numbers by exploring a procedure for adding Magic Squares.

Unifying Ideas: Structure; addition and subtraction.

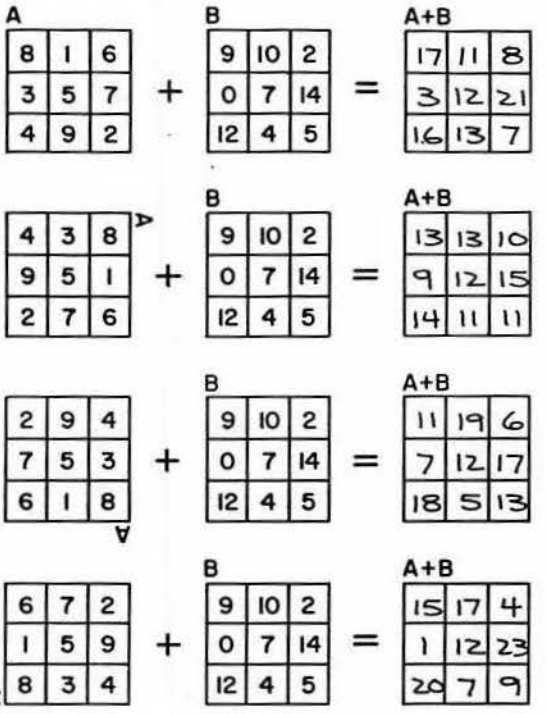
The Lesson: Although pupils are intrigued by Magic Squares, some may be overwhelmed by the difficulties involved in constructing their own. Here are two methods in addition to those presented on the pupil pages.

1. Any four Magic Squares such as A, B, C, and D below can be added in the following way:



2. Several Magic Squares can be constructed from a single pair, A and B, by rotating A one quarter, one half, or three quarters of a revolution clockwise and adding each of these to B as shown at the top of the next column.

Practice sheet 2 provides addition problems of scaled difficulty in which pupils can increase their speed by finding combinations of ten or of a multiple of ten. Such work involves the associative and commutative principles of addition—the idea that in addition grouping and order do not matter. Pupils will also find shortcuts by beginning with the tens or hundreds column instead of with the units column every time.



PS-2

Look for shortcuts!

5	2	4	6	4	5	2	9
5	8	3	9	6	8	6	7
+7	+9	+7	+1	+3	+5	+8	+1
17	19	14	16	13	18	16	17

5	6	5	2	3	15	19	27
5	4	1	1	6	17	16	8
2	3	9	6	7	14	1	19
+8	+2	+3	+8	+4	+15	+6	+12
20	15	18	17	20	61	42	66

50	17	30	10	55	71	43	18
50	40	70	32	20	82	90	12
+12	+60	+28	+90	+80	+30	+16	+34
112	117	128	132	155	183	149	64

30	41	28	95	87	17	79	38
70	60	30	74	3	32	18	23
1	9	2	35	41	89	1	16
+9	+13	+76	+16	+69	+48	+68	+59
110	123	136	220	200	186	166	136

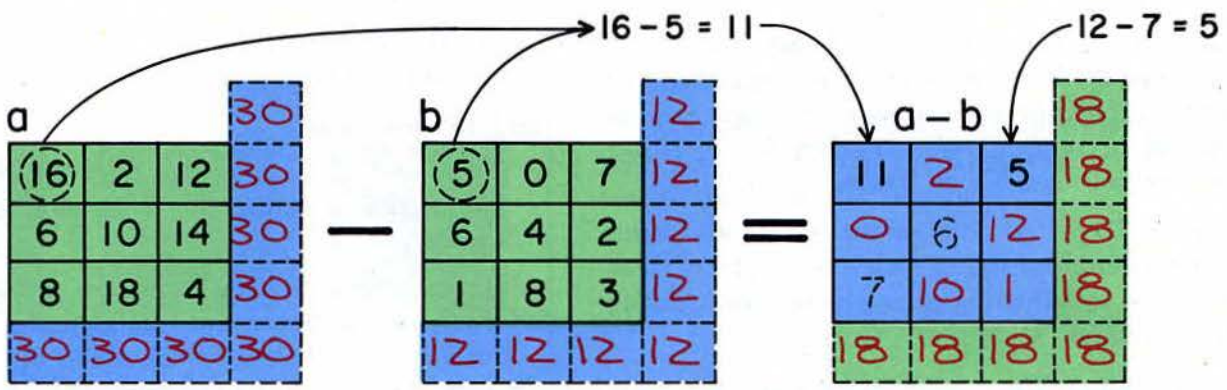
200	126	327	105	836	579
+800	+904	+283	+895	+264	+862
1000	1030	610	1000	1100	1441

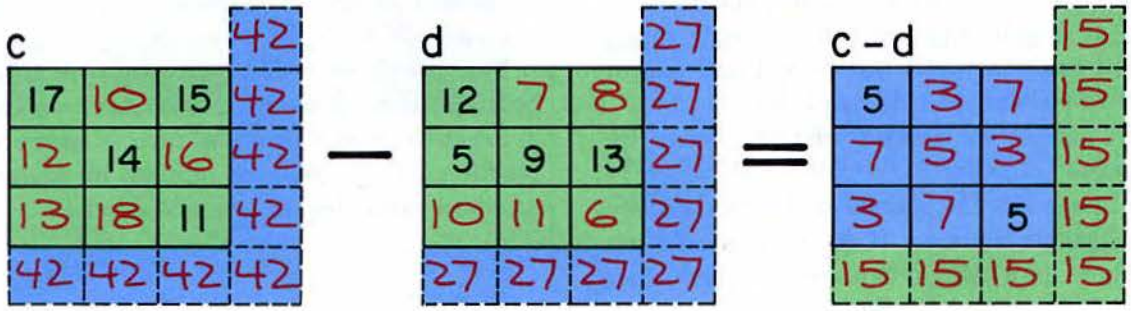
Mary had twice as many books as her sister Jane had. Together they had 12 books. Mary had 8 and Jane had 4 books.

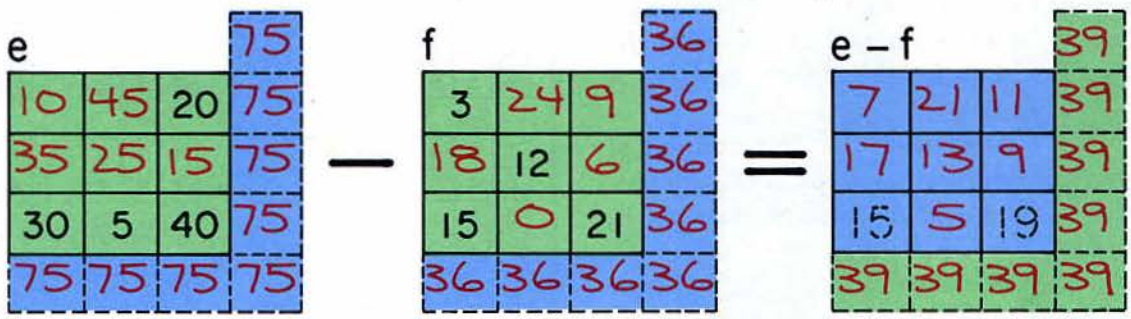
Bill said, "I've got the right change." He gave the clerk 12 coins to pay for a 28¢ purchase. Tell what coins he gave the clerk.
8 PENNIES & 4 NICKELS

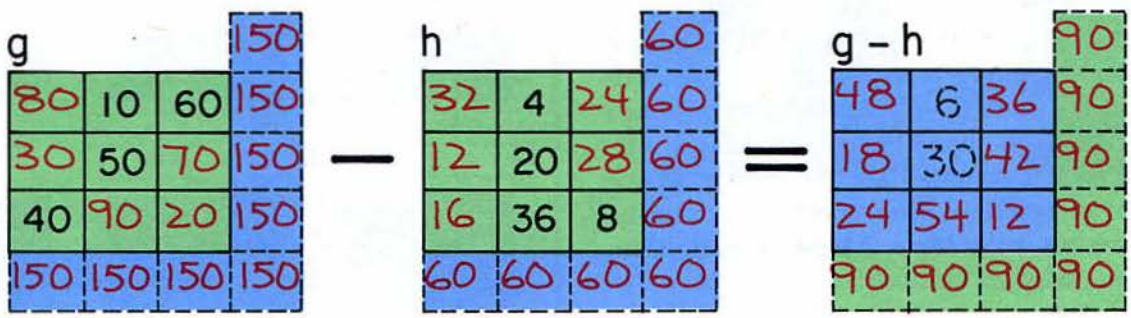
After Ellen had tried some examples, she exclaimed, "I can do the same thing with subtraction. It always works!"

Please finish the examples she gave.









Helen asked, "Does it make any difference in which order you add a pair of Magic Squares? I know that $3 + 9 = 12$ and $9 + 3 = 12$, so I can say that $3 + 9 = 9 + 3$."

Would the same kind of rule be true of Magic Squares?

Let's try an example.

3 Subtraction

Tasks: Pupils apply Ellen's definition of subtraction for Magic Squares and use the clues provided to construct the Magic Squares a *through* h, (a-b), (c-d), (e-f), *and* (g-h).

Purpose: To perfect addition and subtraction skills for whole numbers by exploring a procedure for subtracting Magic Squares.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: On pupil page 3 pupils find that one Magic Square can be subtracted from another—if the pair of squares is carefully selected to avoid the use of negative numbers. The examples on pupil page 3 and practice sheet 3 are, of course, carefully selected. Other examples can be constructed by first adding a pair of Magic Squares:

A				B				A+B		
7	0	5	+	9	10	2	=	16	10	7
2	4	6		0	7	14		2	11	20
3	8	1		12	4	5		15	12	6

This addition process suggests two related subtraction problems:

A+B				A				B		
16	10	7	−	7	0	5	=	9	10	2
2	11	20		2	4	6		0	7	14
15	12	6		3	8	1		12	4	5

A+B				B				A		
16	10	7	−	9	10	2	=	7	0	5
2	11	20		0	7	14		2	4	6
15	12	6		12	4	5		3	8	1

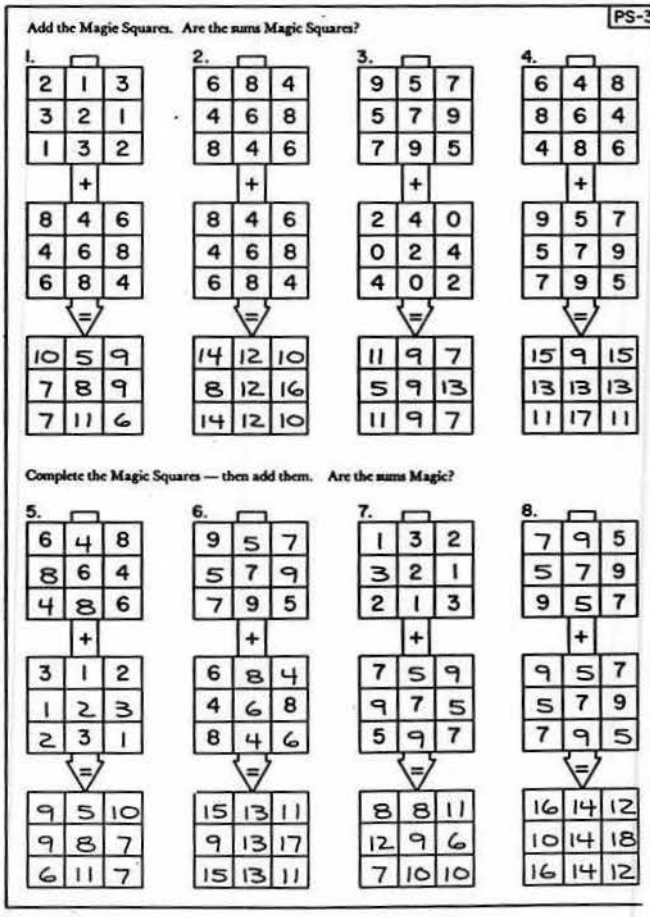
An example that does not work without using negative numbers is considered on pupil page 4.

If pupils are familiar with negative numbers, any Magic Square can be subtracted from any other:

6	7	2		9	2	7		−3	5	−5	−3
1	5	9	−	4	6	8	=	−3	−1	1	−3
8	3	4		5	10	3		3	−7	1	−3
								−3	−3	−3	−3

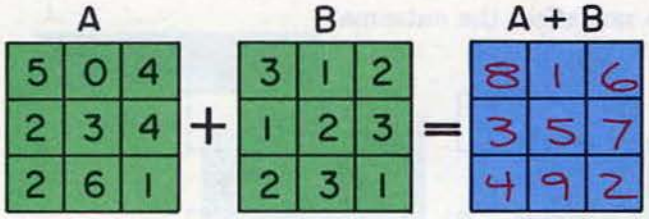
The question asked at the bottom of pupil page 3 leads into the activity presented on pupil page 4.

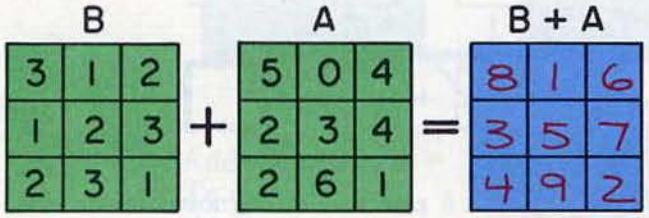
Practice sheet 3 provides practice in adding Magic Squares. In the top section the addition is straightforward. In the bottom section pupils must first reconstruct the Magic Squares from the clues given —and then add them. They should check each completed Magic Square to see if the Magic property is maintained.



Below we can add a pair of Magic Squares in two different ways. Does the same Magic Square result?

I.



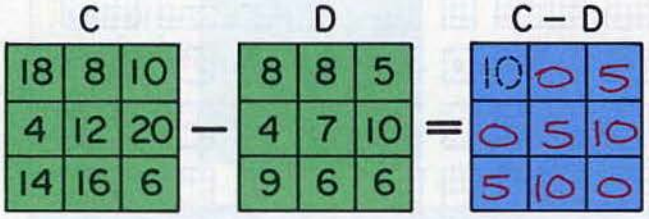


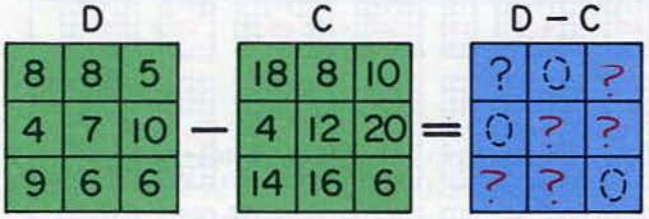
What do you notice about the sums A + B and B + A? Choose any pair of Magic Squares already studied. Add them. Does the order make any difference?

Does the order make any difference in subtraction?

Here is an example to try:

II.





Clearly, the rule does not hold for subtraction, but you expected that. You know that

$$3 + 7 = 7 + 3$$

$$9 + 5 = 5 + 9$$

etc.

(Mathematicians refer to the above sentences as instances of the Commutative Principle for Addition.)

You also know that

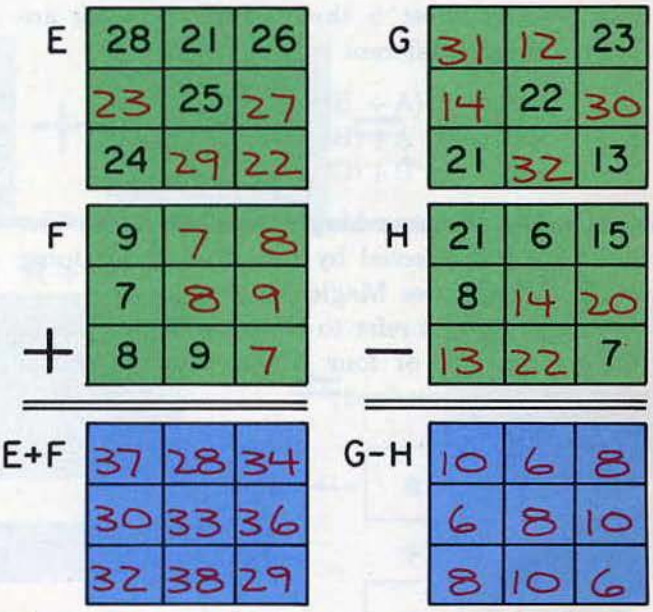
$$8 - 5 \neq 5 - 8$$

$$13 - 6 \neq 6 - 13$$

etc.

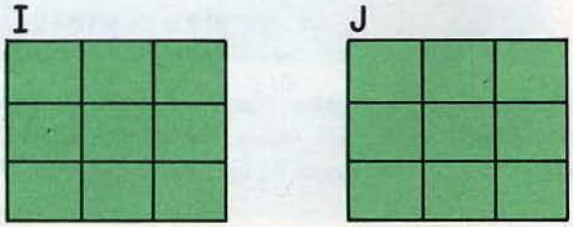
We can also arrange Magic Squares in a vertical form. Here is an example of addition and an example of subtraction of Magic Squares. Please complete them.

III.



IV.

Make two Magic Squares that are different from any you have studied thus far.



Tasks: In exercises I and II pupils test addition and subtraction of Magic Squares for commutativity. In exercise III pupils add and subtract Magic Squares arranged in vertical form. In exercise IV pupils construct their own Magic Squares.

Purpose: To perfect addition and subtraction skills for whole numbers and to review principles of these operations by exploring principles of operations on Magic Squares.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: Pupils should by now suspect that the arithmetic of Magic Squares behaves very much like the arithmetic of whole numbers. Pupil page 4 is designed to strengthen this suspicion.

On pupil page 1 we defined what we mean by Magic Squares (of order 3).

On pupil page 2 we explained what we mean by addition of Magic Squares: If A and B are Magic Squares, there is a Magic Square C called their sum. In short, for any Magic Squares A and B, there is a Magic Square C such that $A+B=C$.

On pupil page 4 we notice that the order in which we add a pair of Magic Squares does not affect the outcome. We say that for any Magic Squares A and B, $A+B=B+A$.

On pupil page 5 we shall see that the grouping of Magic Squares in addition does not affect the outcome. We say that for any Magic Squares A, B, and C, $(A+B)+C=A+(B+C)$.

On pupil page 3 we defined subtraction of Magic Squares and noted that it is not always possible to subtract one from another unless negative numbers are available. Therefore, we can only say that when Magic Square B can be subtracted from Magic Square C the outcome is a Magic Square. If we call this difference A then $C-B=A$. Moreover, we can say that if $A+B=C$ then $C-B=A$ and $C-A=B$.

On pupil page 4 we note that the order of Magic Squares in subtraction does affect the outcome except when we subtract a Magic Square from itself. In summary,

$A+B=C$
[addition of Magic Squares is defined]

$A+B=B+A$
[the addition is commutative]

$(A+B)+C=A+(B+C)$
[the addition is associative]

and, in some cases

$C-B=A$
[subtraction is defined for some pairs of Magic Squares]

All these properties closely resemble properties for addition and subtraction of whole numbers:

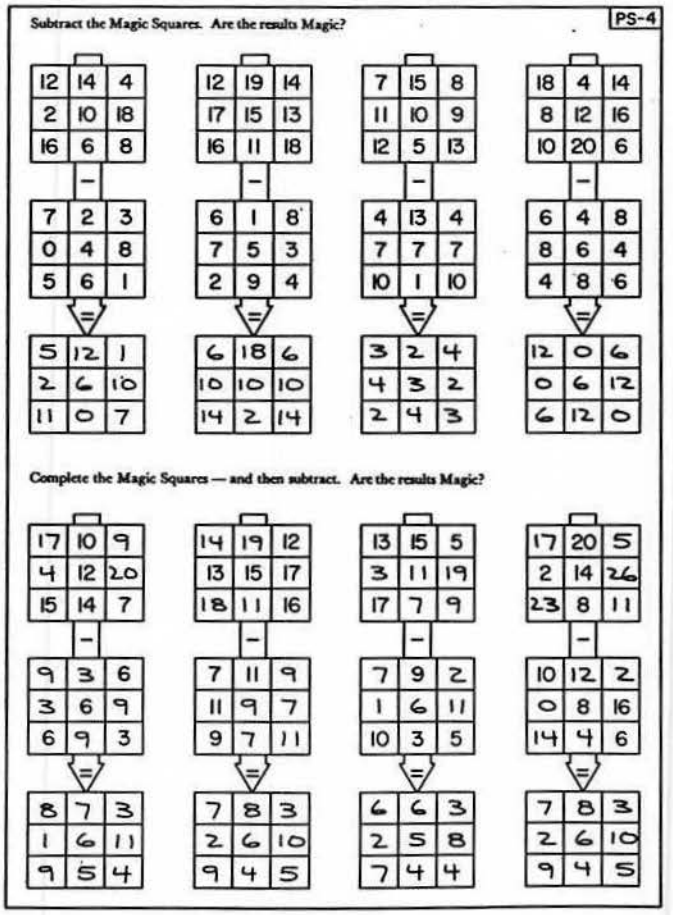
For any two whole numbers a and b, there is a whole number c called their sum and $a+b=c$.

For any two whole numbers a and b, $a+b=b+a$.

For any three whole numbers a, b, and c, $(a+b)+c=a+(b+c)$.

We note further that, for any three whole numbers a, b, and c, if $a+b=c$ then $c-b=a$ and $c-a=b$.

Practice sheet 4 provides further practice in subtracting Magic Squares similar in design to the addition examples on practice sheet 3.


Subtract the Magic Squares. Are the results Magic?
PS-4
Complete the Magic Squares — and then subtract. Are the results Magic?

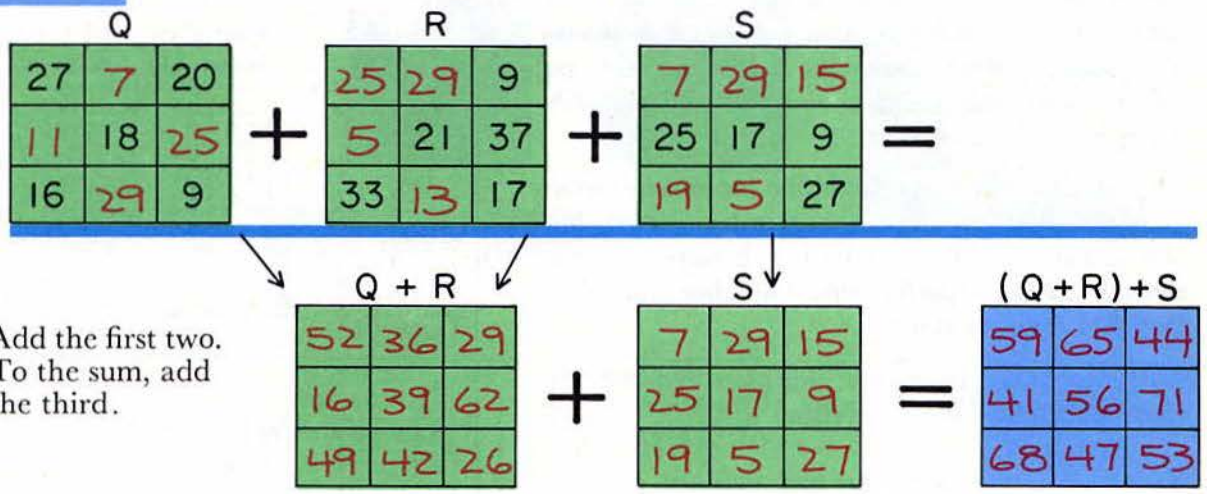
Warren asked whether we might not be able to add three Magic Squares. "Something like adding three numbers," he explained. "I wonder if the grouping would make any difference?"

What do you think?

Here is an experiment.

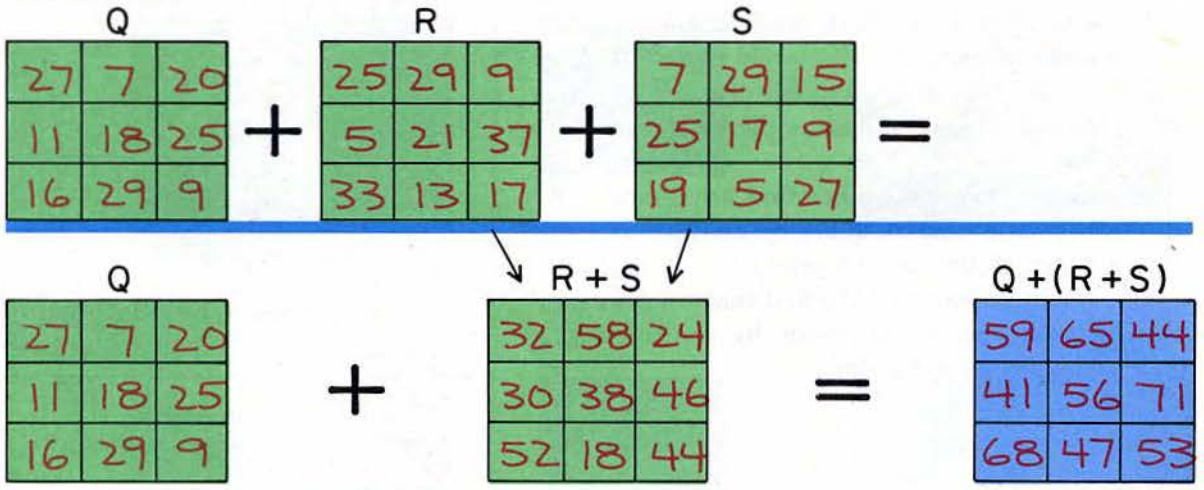
I.

In one grouping



II.

In another grouping



Is (Q + R) + S the same as Q + (R + S)?

III.

Try this same grouping experiment with other groups of 3 Magic Squares. Choose some we have already considered or make up some of your own.

Our results remind us of what mathematicians call the Associative Principle for Addition.

Here are two instances of this principle in adding numbers.

$$(8 + 7) + 3 = 8 + (7 + 3)$$
$$15 + 3 = 8 + 10$$

and

$$(10 + 7) + 9 = 10 + (7 + 9)$$
$$17 + 9 = 10 + 16$$

5 Addition

Tasks: In exercises I and II pupils test the associative principle for addition of Magic Squares. In exercise III pupils select three Magic Squares other than Q, R, and S to test this principle again.

Purpose: To perfect addition and subtraction skills for whole numbers and to review principles of these operations by exploring principles of operations on Magic Squares.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: The following sentence summarizes an important principle for addition of Magic Squares:

$$(Q+R)+S=Q+(R+S)$$

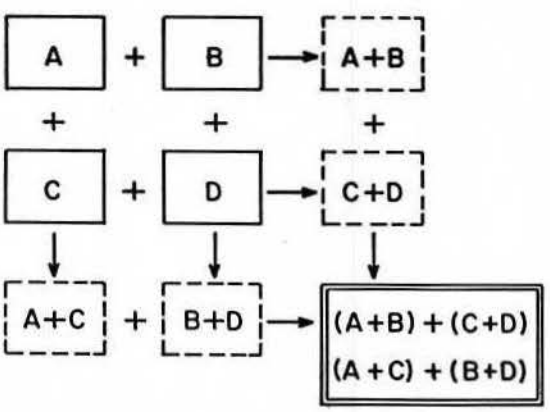
The expression on the left indicates that S is to be added to the sum of Q and R; the one on the right indicates that the sum of R and S is to be added to Q. Pupil page 5 shows that the same Magic Square results from both of these addition problems.

On practice sheet 5 three Magic Squares are added in several different ways:

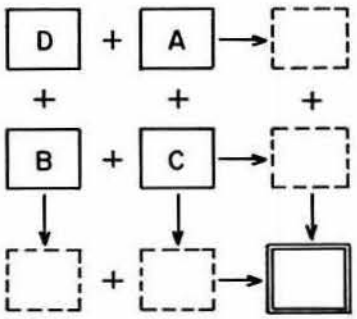
$$(A+B)+C$$
$$A+(B+C)$$
$$B+(C+A)$$

In all cases the same Magic Square results. The outcome is not affected by the order or grouping used in adding three Magic Squares.

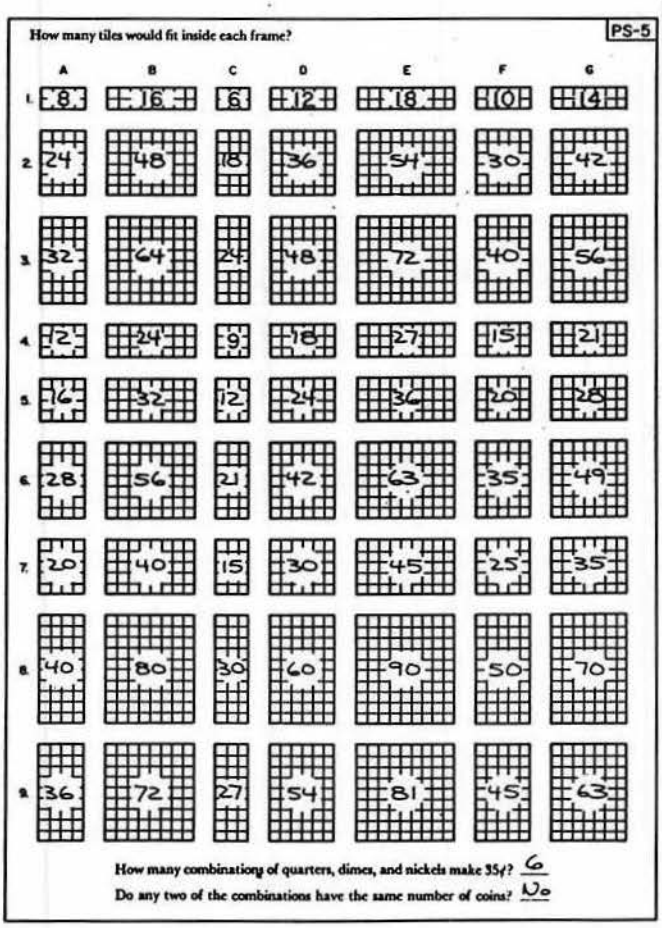
You may wish to refer to Notes on Pupil Page 2. The arrangement of four Magic Squares shown there can be summarized:



Once again we notice that different orders and groupings of Magic Squares lead to the same result. We can use the same Magic Squares in different arrangements to provide more evidence that the order and grouping of Magic Squares in addition do not affect the outcome.



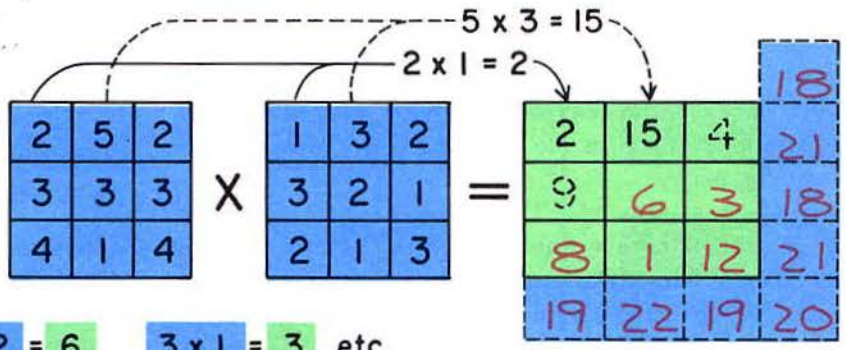
Practice sheet 5 provides a diagnostic check on pupil mastery of the basic facts of multiplication. Those who know them thoroughly will count the number of rows and columns in each array and write the product of these numbers in the grid. The unsure can count their way to a correct answer.


How many tiles would fit inside each frame?
PS-5
A B C D E F G
1. 8 16 6 12 18 10 14
2. 24 48 18 36 54 30 42
3. 32 64 24 48 72 40 56
4. 12 24 9 18 27 15 21
5. 16 32 12 24 36 20 28
6. 28 56 21 42 63 35 49
7. 20 40 15 30 45 25 35
8. 40 80 30 60 90 50 70
9. 36 72 27 54 81 45 63
How many combinations of quarters, dimes, and nickels make 35¢? 6
Do any two of the combinations have the same number of coins? No

"How about multiplication?" Sarah asked. "We can always add Magic Squares and sometimes we can subtract one from another. Can we multiply one Magic Square by another?"

What do you think?

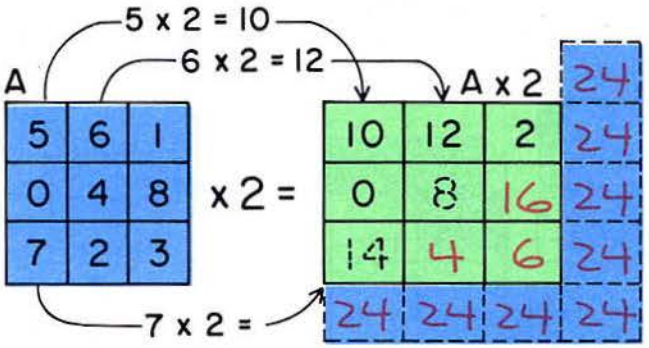
Suppose we try a plan similar to our plan in adding Magic Squares. Finish the work started on the right:



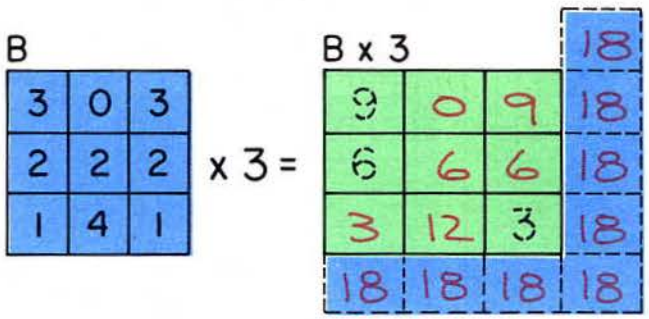
2x2 = 4 3x3 = 9 3x2 = 6 3x1 = 3 etc.

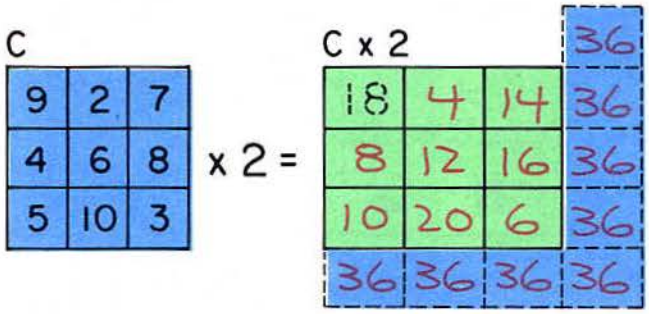
Is the result a Magic Square? No.

Let's try another plan. Perhaps we can multiply a Magic Square by a number in this way:

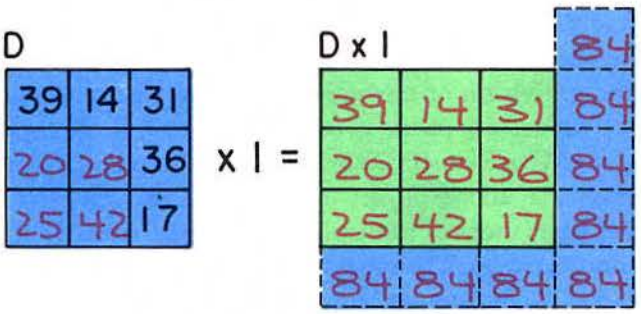


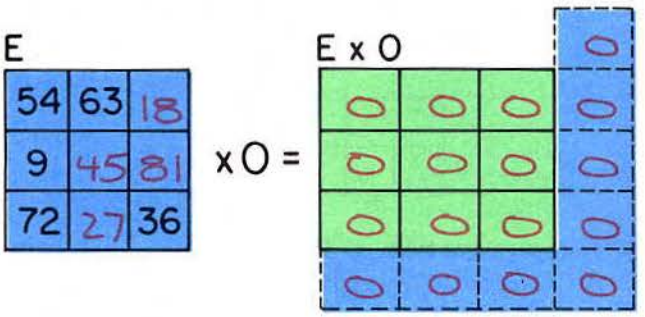
That's encouraging. Let's try some more.

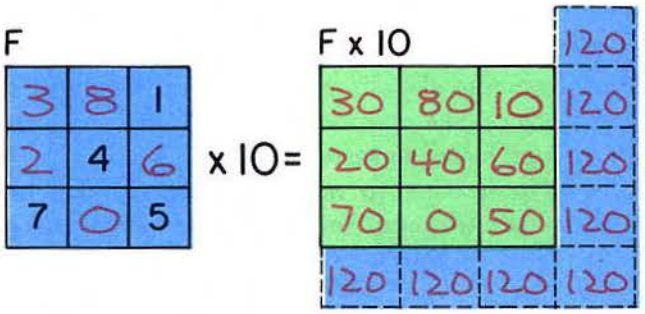




What happens when we multiply Magic Squares by 1? By 0? By 10?







6 Multiplication

Tasks: For the exercise at the top of pupil page 6 pupils carry out the multiplication as indicated to determine whether or not the product of the pair of Magic Squares is a Magic Square. In exercises A-C pupils use the clues provided to determine whether or not the product of a Magic Square and a whole number is a Magic Square. In exercises D-F pupils carry out the indicated multiplications to answer the questions asked before exercise D.

Purpose: To establish a definition for multiplication of Magic Squares and whole numbers; to explore the properties of 0, 1, and 10 with respect to this multiplication; to perfect whole-number multiplication and division skills.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: We hope the first activity on pupil page 6 will deliver a solid jolt to your pupils! What about multiplication? Everything went smoothly with addition of Magic Squares.

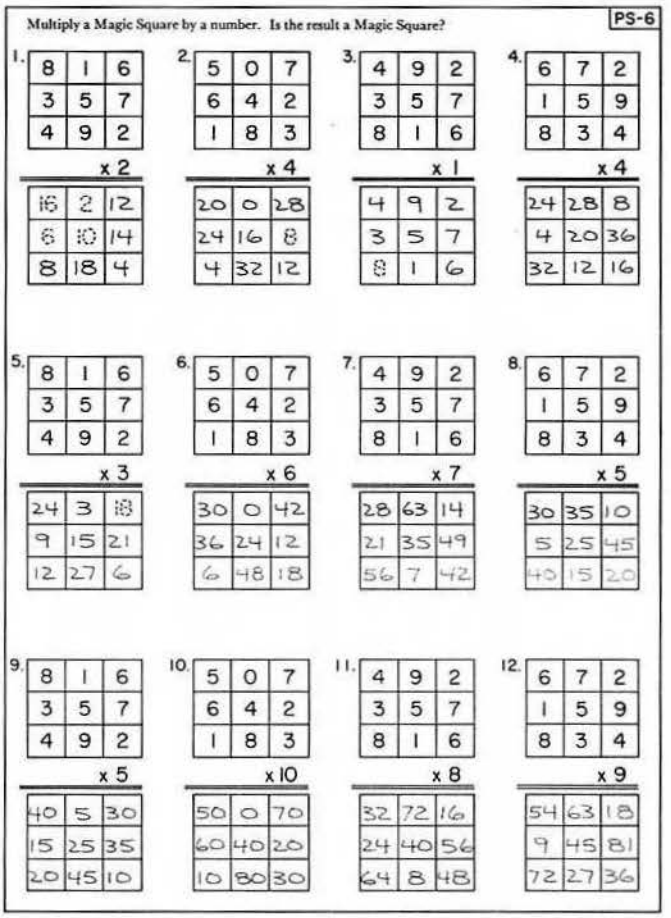
What can we do to make multiplication behave as we would anticipate? We would expect that we can find a way to multiply a pair of Magic Squares and obtain a Magic Square as the product.

Not so!

Try as we may, we cannot find any "nice" way to multiply one Magic Square by another that will give a Magic Square as the product.

But we find a way out! We find that we can easily explain what we might mean by multiplying a Magic Square by a *number*.

Examples on pupil page 6 and on practice sheet 6 explore this idea. (Note that since all basic multiplication facts are included in practice sheet 6 it can be used as a checkup on pupil mastery of these facts.)



Finish each pair of blue-tinted boxes (A-D) to make Magic Squares.

Then add the squares in each pair to make a new one.

Summary: The sum of Magic Squares is (always, sometimes, never) a Magic Square.

☑ ☐ ☐

A.

8	1	6
3	5	7
4	9	2

+

25	4	19
10	16	22
13	28	7

=

33	5	25
13	21	29
17	37	9

B.

17	1	15
9	11	13
7	21	5

+

7	38	15
28	20	12
25	2	33

=

24	39	30
37	31	25
32	23	38

C.

6	27	12
21	15	9
18	3	24

+

14	63	28
49	35	21
42	7	56

=

20	90	40
70	50	30
60	10	80

D.

11	13	3
1	9	17
15	5	7

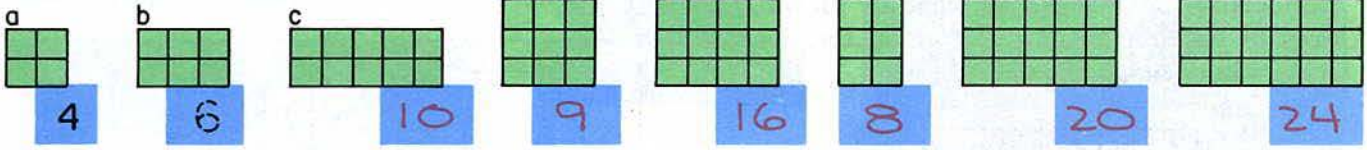
+

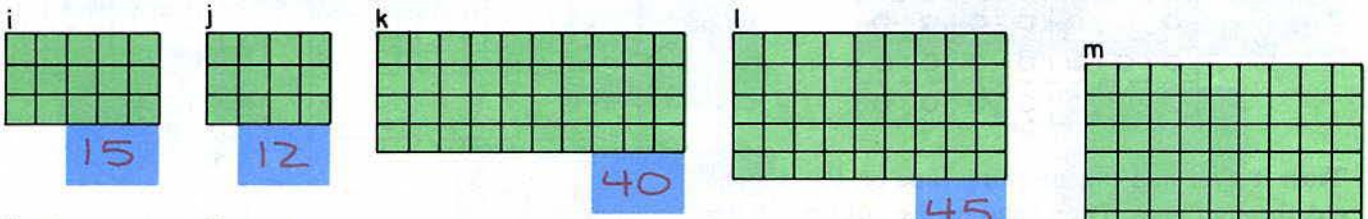
8	6	16
18	10	2
4	14	12

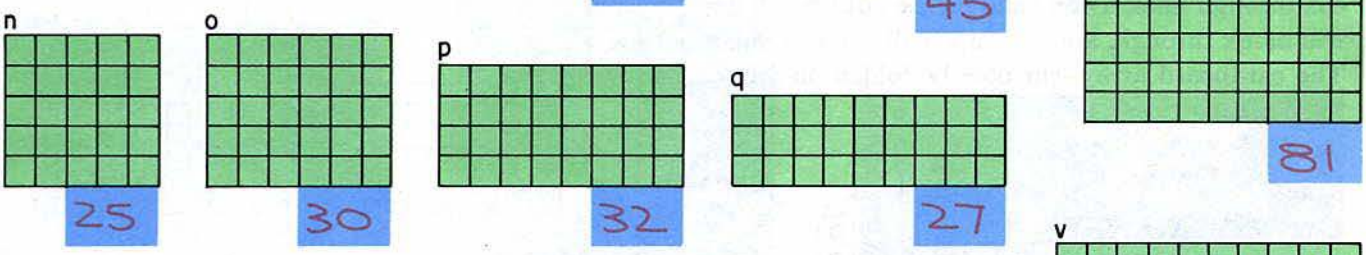
=

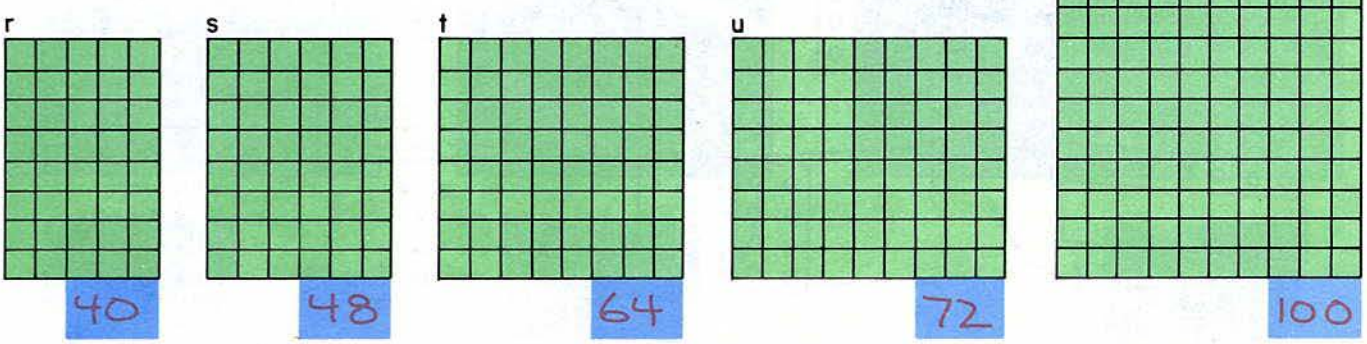
19	19	19
19	19	19
19	19	19

What's My Rule?









7 Addition of magic squares and display of multiplication facts

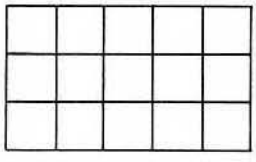
Tasks: *In exercises A-D pupils complete each pair of Magic Squares and compute their sum. Pupils then place a check in the appropriate box to make the summary statement true. In exercises* a-v *pupils use the clues provided to determine the rule and make an appropriate entry in the blue space below each rectangular array.*

Purpose: To review another property (closure) of addition of Magic Squares while perfecting addition and subtraction skills for whole numbers; to review the basic multiplication and division facts (and fact teams) for whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: Addition and subtraction of Magic Squares are reviewed on pupil pages 7 and 8 respectively. Multiplication of a Magic Square by a number is reviewed on pupil page 9. Pupil page 10 introduces division of a Magic Square by a number.

The bottom sections of pupil pages 7 through 10 and practice sheets 7 through 12 are included especially for those pupils who need more work with the basic facts of multiplication (and their division counterparts). The representations selected (various forms of rectangular arrays) provide an easy way of visualizing multiplication and division facts. We assume that pupils understand the sketch:

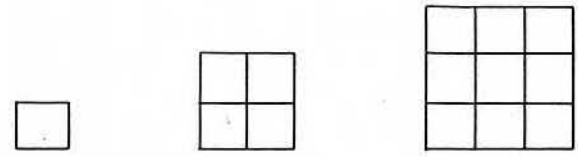


as a representation of four closely related statements:

$5 \times 3 = 15$ $\quad$ $15 \div 3 = 5$
$3 \times 5 = 15$ $\quad$ $15 \div 5 = 3$

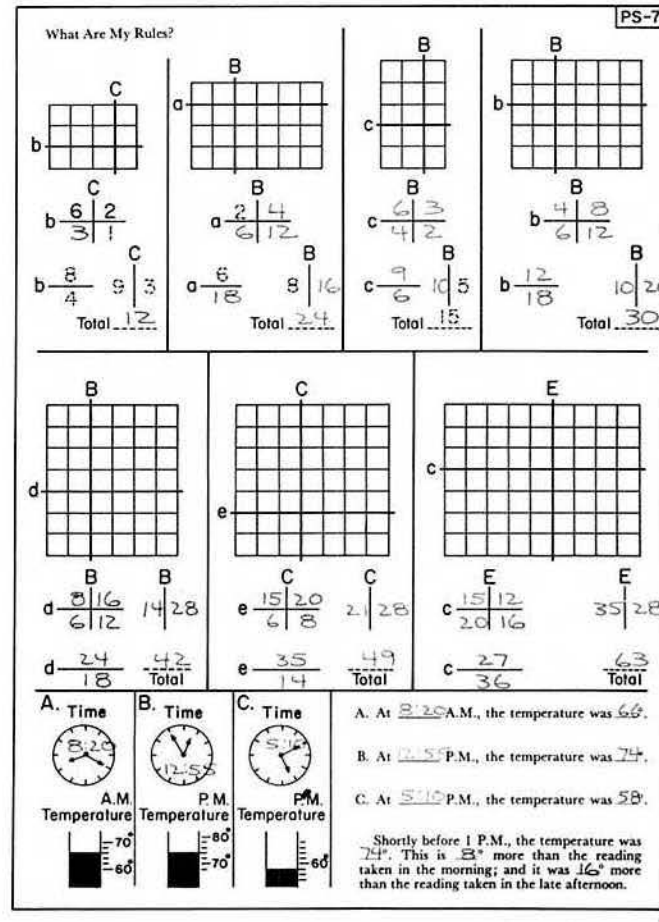
If they have difficulty constructing the entire fact team from the sketch, you may find it helpful to make several sketches on the board and write the entire fact team suggested by each.

It is interesting to note that certain sketches suggest fact teams of fewer statements:



$1 \times 1 = 1$ $\quad$ $2 \times 2 = 4$ $\quad$ $3 \times 3 = 9$
$1 \div 1 = 1$ $\quad$ $4 \div 2 = 2$ $\quad$ $9 \div 3 = 3$

The numbers of squares in such arrays—1, 4, 9, 16, 25, 36, etc., together with 0—are square numbers. Some pupils may wish to extend this list of square numbers far beyond 100 (10×10). . . 121, 144, 169, etc. (If they do, suggest that they look for one member in the list that is twice as large as another in the list. In the list of the first 100 square numbers, $70 \times 70 = 4900$ and $99 \times 99 = 9801$ is as close as they can come. If they go on they will have no more success! Such experiments performed now may give more meaning to the explanation later on that no such pair of square numbers exists.


What Are My Rules?
PS-7
Total 12
Total 24
Total 15
Total 30
Total
Total
A. Time
B. Time
C. Time
A.M. Temperature
P.M. Temperature
P.M. Temperature
A. At 8:20 A.M., the temperature was 66°.
B. At 12:55 P.M., the temperature was 74°.
C. At 5:10 P.M., the temperature was 58°.
Shortly before 1 P.M., the temperature was 74°. This is 8° more than the reading taken in the morning; and it was 16° more than the reading taken in the late afternoon.

Finish each pair of green-tinted boxes to make Magic Squares.

Then subtract the second square from the top square to make a new one.

Summary: The difference of a pair of Magic Squares is (always, sometimes, never) a Magic Square.

☐ ☑ ☐

A.

27	6	21
12	18	24
15	30	9

−

15	1	11
5	9	13
7	17	3

=

12	5	10
7	9	11
8	13	6

B.

40	0	26
8	22	36
18	44	4

−

49	0	35
14	28	42
21	56	7

=

?	0	?
?	?	?
?	?	?

C.

50	57	22
15	43	71
64	29	36

−

36	42	12
6	30	54
48	18	24

=

14	15	10
9	13	17
16	11	12

D.

27	6	21
12	18	24
15	30	9

−

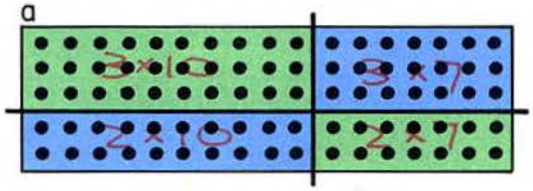
24	79	62
93	55	17
48	31	86

=

3	?	?
?	?	7
?	?	?

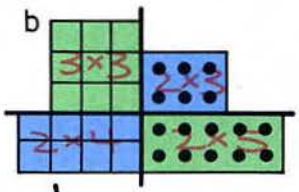
What's My Rule?

a



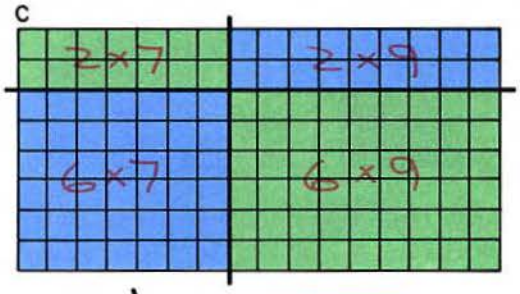
30	21
20	14

b



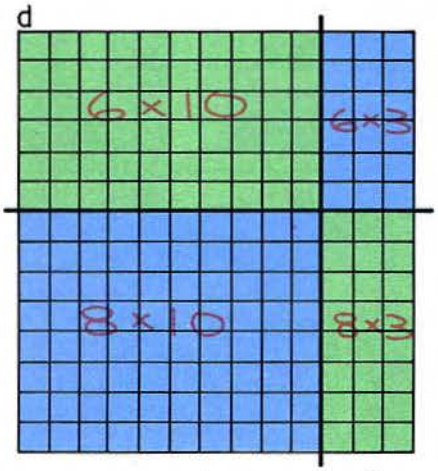
9	6
8	10

c



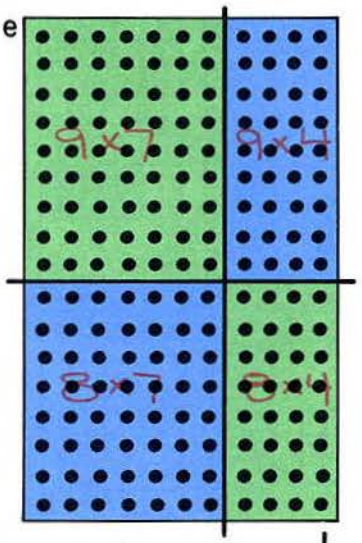
14	18
42	54

d



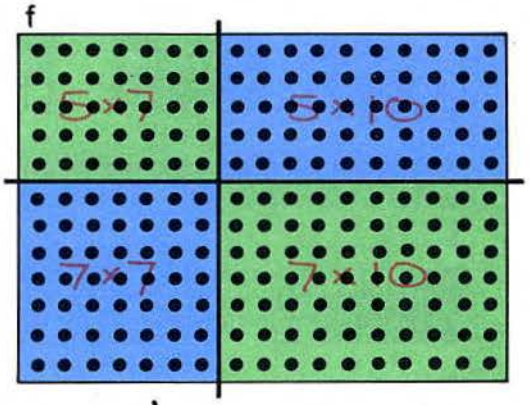
60	18
80	24

e



63	36
56	32

f



35	50
49	70

8 Subtraction of magic squares and display of multiplication facts

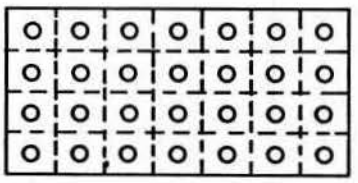
Tasks: For exercises A-D pupils complete each pair of Magic Squares and (when possible) compute the difference for each pair. For exercises a-f pupils determine the rule and make appropriate entries below each array. Pupils may also need or be asked to record the expression of the product of the factors on the arrays.

Purpose: To review subtraction of Magic Squares; to review the rationale for a multiplication algorithm for whole numbers.

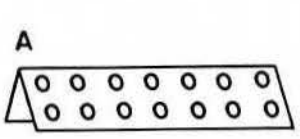
Unifying Ideas: Structure; addition and subtraction; multiplication and division.

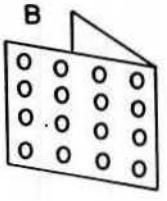
The Lesson: In exercises A-D pupils discover that subtraction of Magic Squares, like subtraction of numbers, is not always possible (without negative numbers). Each of the squares contains enough clues to make completion simply practice in computation.

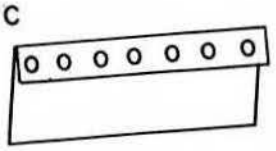
The arrays on pupil page 8 and practice sheet 8 provide a lot of practice in little space. Similar manipulative devices can be made by drawing an array of dots on cardboard. Cut on the lines as shown—but not all the way through.

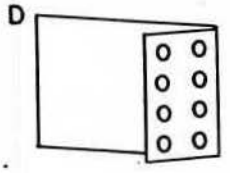


Turn it over and put masking tape on the almost-cut-through lines. Then bend on the cut lines. They will break through, and the tape will act as a hinge. The cardboard array can now be folded on any of the dividing lines.







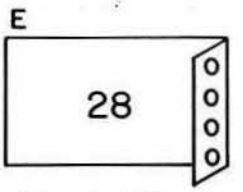


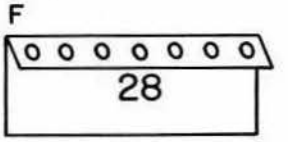
How many dots do you see?

I'm looking at the part that's folded back—the part you can't see. How many dots do I see?

Now the pupils will get excited. They are confronted with a problem they must think through.

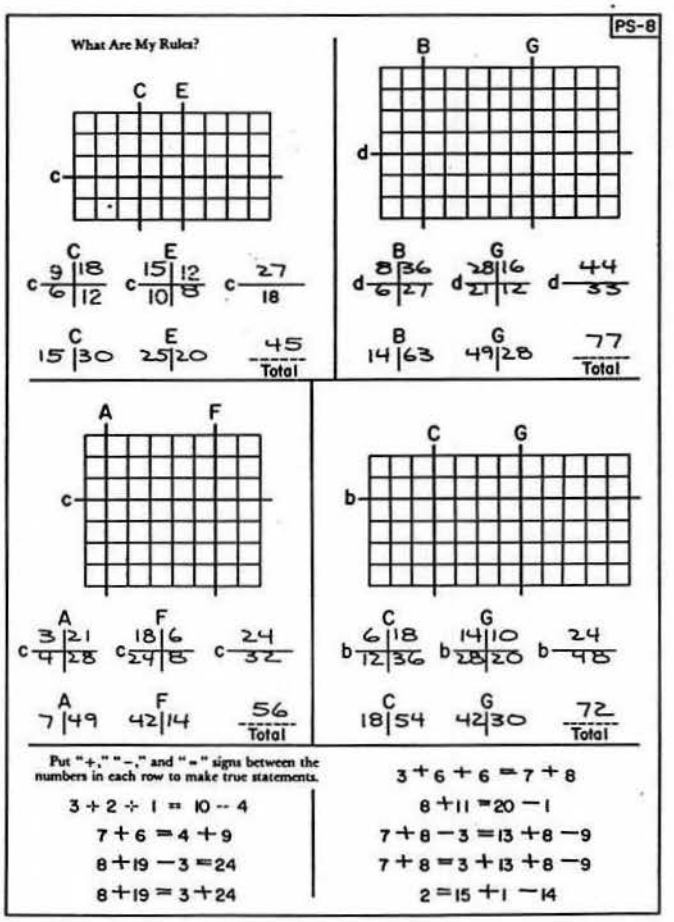
With only a few such folding arrays, all the basic facts of multiplication can be indicated. You may wish to write the total number of dots on the back and display the cards as shown:





For E, ask: How many in each row on the entire card? For F: How many in each column on the entire card? (28 ÷ 4 = ____, 28 ÷ 7 = ____)

Before pupils complete the exercises at the bottom of practice sheet 8, review the section in the Introduction on punctuation and the order of operations (pages 45-46).



Summary:

The product of a Magic Square by a number is (always, sometimes, never) a Magic Square.

✓ ☐ ☐

1.

8	9	4
3	7	11
10	5	6

x 3 =

24	27	12
9	21	33
30	15	18

2.

7	2	9
8	6	4
3	10	5

x 4 =

28	8	36
32	24	16
12	40	20

3.

1	6	5
8	4	0
3	2	7

x 6 =

6	36	30
48	24	0
18	12	42

4.

10	45	20
35	25	15
30	5	40

x 2 =

20	90	40
70	50	30
60	10	80

5.

16	2	12
6	10	14
8	18	4

x 5 =

80	10	60
30	50	70
40	90	20

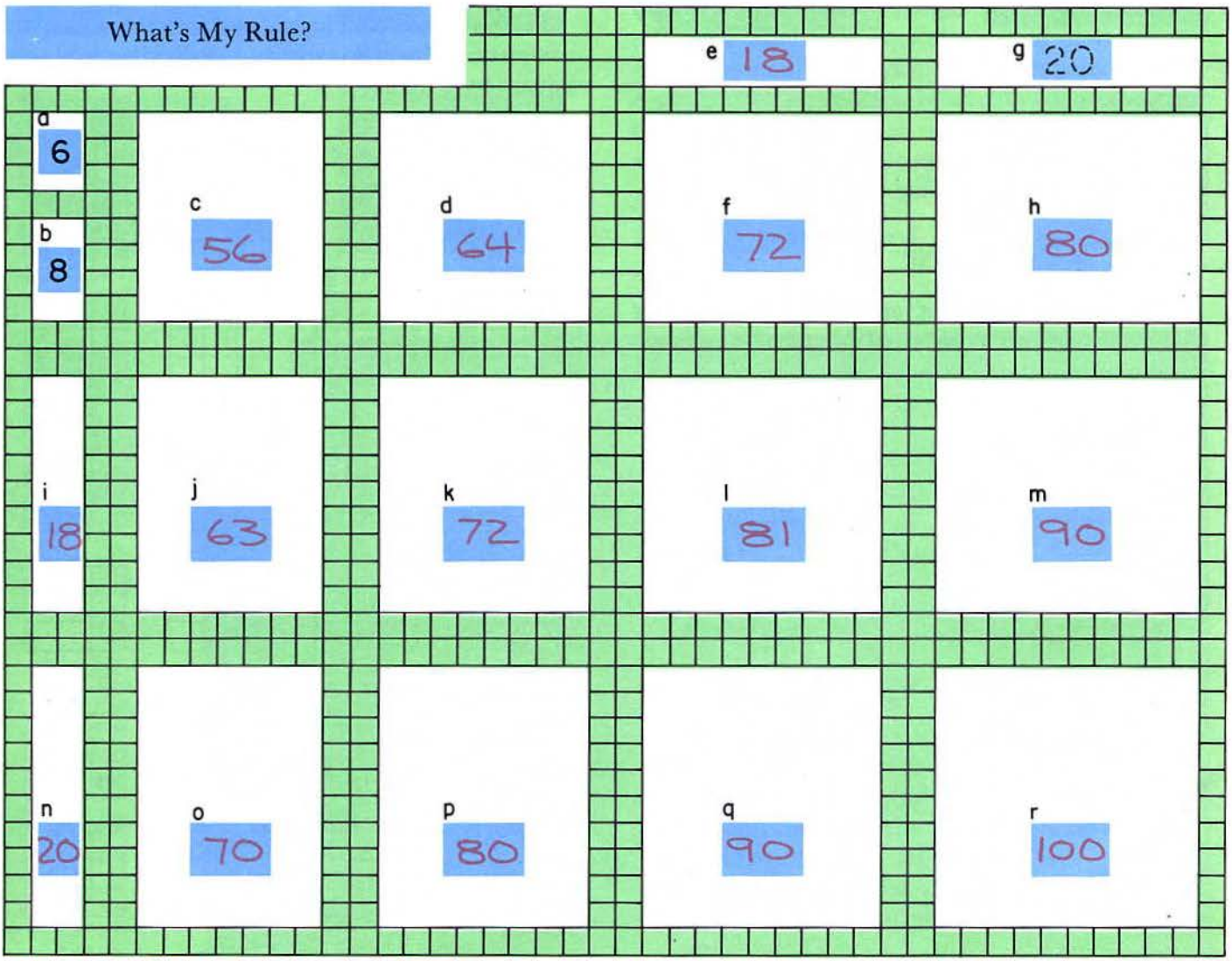
6.

3	8	1
2	4	6
7	0	5

x 7 =

21	56	7
14	28	42
49	0	35

What's My Rule?



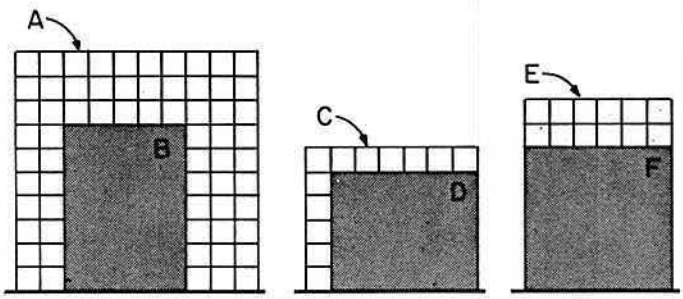
9 Multiplication of magic squares and display of multiplication facts

Tasks: *In exercises 1-6 pupils carry out the indicated multiplication to construct the product. In exercises* a-r *pupils use the clues provided to determine the rule and make appropriate entries in the blue tinted space.*

Purpose: To review the fact that multiplying a Magic Square by a number results in a Magic Square; to review multiplication facts.

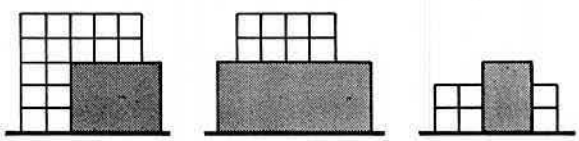
Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: A set of cards based on the sketches in the *What's My Rule?* section of pupil page 9 may be used to help pupils who still have trouble with multiplication combinations. They can be used as flash cards, or, for a richer activity, place one face down on another:

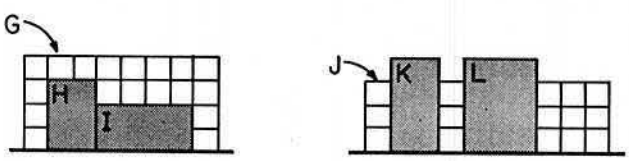


Ask: How many on A? On B? How many of the blocks on A can you see? How many are hidden? Does B cover more or less than half of A?

The examples below require a bit of judgment. Pupils may want to use the front of another card to measure the back of a card. Such a practical approach should be commended.



More complicated arrangements can be used:



How many behind H? I? K? L? How many tiles on G? On J? How many tiles of G can you see? Of J? How many tiles of G are covered? Of J? How many tiles of J are covered by K? By L?

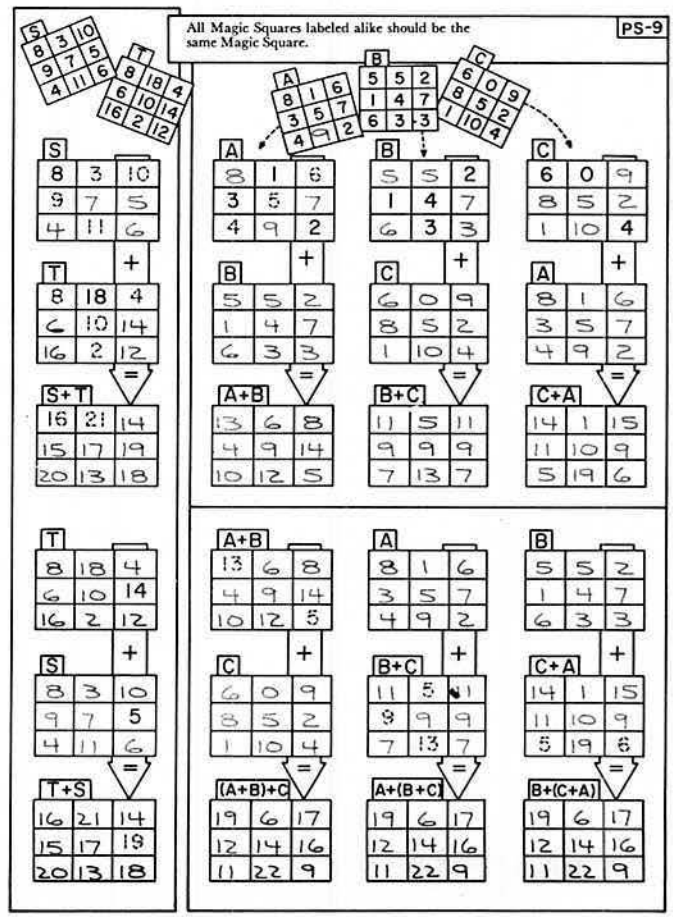
Have pupils name some of the numbers that can be represented by cards having different shapes.

12 . . . 2 by 6, 3 by 4, 1 by 12
18 . . . 2 by 9, 3 by 6, 1 by 18

Have pupils complete exercises highlighting the inverse relationship between multiplication and division. They may need to be reminded that in multiplication we have two factors and find a missing product, whereas in division we have the product and one factor and find a missing factor.

Multiplication	*Division*
a. $2 \times 8 =$ ____	b. $2 \times$ ____ $= 16$
c. $2\overline{)8}$	d. $2\overline{)16}$
e. $3\overline{)5}$	f. $\overline{)15}5$

In completing practice sheet 9 pupils will find that order and grouping do not affect the sum of two or three Magic Squares.



A Summary:

The quotient of a Magic Square by a number is (always, sometimes, never) a Magic Square.

☐ ☐ ☐

1.

6	20	10
16	12	8
14	4	18

÷2=

3	10	5
8	6	4
7	2	9

3.

27	6	21
12	18	24
15	30	9

÷3=

9	2	7
4	6	8
5	10	3

2.

24	28	8
4	20	36
32	12	16

÷4=

6	7	2
1	5	9
8	3	4

4.

15	1	11
5	9	13
7	17	3

÷10=

?	?	?
?	?	?
?	?	?

B Can you rebuild Magic Squares in the green boxes?

1.

15	40	35
50	30	10
25	20	45

÷5=

3	8	7
10	6	2
5	4	9

3.

42	12	36
24	30	36
24	48	18

÷6=

7	2	6
4	5	6
4	8	3

2.

56	0	49
28	35	42
21	70	14

÷7=

8	0	7
4	5	6
3	10	2

4.

56	0	40
16	32	48
24	64	8

÷8=

7	0	5
2	4	6
3	8	1

C

"What's My Rule?"

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Tasks: *In exercise A pupils attempt the indicated divisions to determine whether quotients can be obtained and whether such quotients are Magic Squares. In exercise B pupils reconstruct Magic Squares by "undoing" divisions. In exercise C pupils complete the multiplication table.*

Purpose: To indicate the limitations of division of Magic Squares; to perfect addition, subtraction, multiplication and division skills for whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: After working examples 1-4 of pupil page 10, most pupils will conclude that the result of dividing a Magic Square by a number is only sometimes a Magic Square. Some, however, may use fractions in example 4 and conclude, correctly, that dividing a Magic Square by a (nonzero) number always results in a Magic Square.

Before assigning this page, you may need to explain that in division we often call the given factor the divisor and the missing factor the quotient.

factor or divisor ⟶ 5) 15 ⟵ 3 ← factor or quotient

One way to construct examples of division of Magic Squares so as to avoid fractions is to begin with multiplication examples:

C

6	7	2
1	5	9
8	3	4

× 3 =

(C × 3)

18	21	6
3	15	27
24	9	12

(C × 3)

18	21	6
3	15	27
24	9	12

÷ 3 =

C

6	7	2
1	5	9
8	3	4

Have pupils find the upper left to lower right diagonal in the table at the bottom of pupil page 10. It contains 0, 1, 4, 9, 16, 25, 36, 49, 64, and 81—the square numbers. The table is symmetric about this diagonal; each entry on one side of it is repeated in the symmetric position on the other side (showing the commutative principle of multiplication for whole numbers).

What numbers other than 0 are repeated three or more times in the body of the table? The smallest is 4: 1×4, 4×1, 2×2.

Have more able pupils consider the upper right to lower left diagonal and notice the difference in successive terms:

0 8 14 18 20 20 18 14 8 0

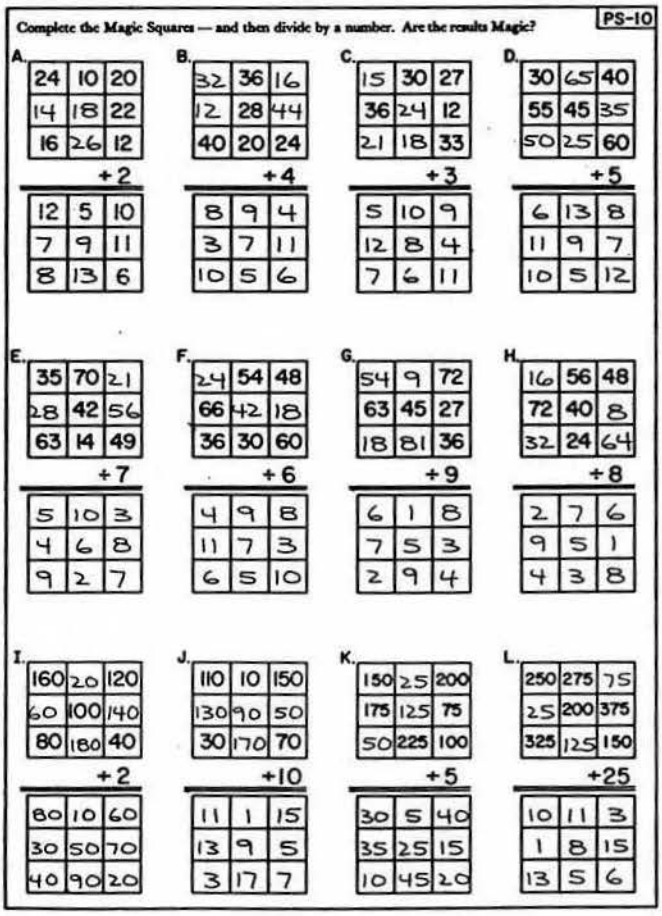
8 6 4 2 0 2 4 6 8

Have them do the same for the numbers that parallel either diagonal. For example, the numbers two to the right of the diagonal of square numbers are:

0 3 8 15 24 35 48 63

3 5 7 9 11 13 15

Practice sheet 10 provides further practice in completing Magic Squares and in dividing a Magic Square by a number.



Complete the Magic Squares — and then divide by a number. Are the results Magic? PS-10

A.

24	10	20
14	18	22
16	26	12

+2

12	5	10
7	9	11
8	13	6

B.

32	36	16
12	28	44
40	20	24

+4

8	9	4
3	7	11
10	5	6

C.

15	30	27
36	24	12
21	18	33

+3

5	10	9
12	8	4
7	6	11

D.

30	65	40
55	45	35
50	25	60

+5

6	13	8
11	9	7
10	5	12

E.

35	70	21
28	42	56
63	14	49

+7

5	10	3
4	6	8
9	2	7

F.

24	54	48
66	42	18
36	30	60

+6

4	9	8
11	7	3
6	5	10

G.

54	9	72
63	45	27
18	81	36

+9

6	1	8
7	5	3
2	9	4

H.

16	56	48
72	40	8
32	24	64

+8

2	7	6
9	5	1
4	3	8

I.

160	20	120
60	100	140
80	180	40

+2

80	10	60
30	50	70
40	90	20

J.

110	10	150
130	90	50
30	170	70

+10

11	1	15
13	9	5
3	17	7

K.

150	25	200
175	125	75
50	225	100

+5

30	5	40
35	25	15
10	45	20

L.

250	275	75
25	200	375
325	125	150

+25

10	11	3
1	8	15
13	5	6

Following Directions

A.

	3	15	28	44	19
+ 5	8	20	33	49	24

B.

	7	11	20	28	52
− 3	4	8	17	25	49

C.

	2	5	9	0	10
x 3	6	15	27	0	30

D.

	8	10	18	20	50
÷ 2	4	5	9	10	25

E.

	12	21	47	65	99
− 8	4	13	39	57	91

F.

	6	23	17	85	49
+ 9	15	32	26	94	58

G.

	3	5	7	9	6
x 7	21	35	49	63	42

H.

	8	12	19	49	88
+12	20	24	31	61	100

I.

	2	3	5	8	0
x 9	18	27	45	72	0

J.

	24	6	30	60	48
÷ 6	4	1	5	10	8

K.

	16	28	20
+ 3	19	31	23
x 2	32	56	40
÷ 4	4	7	5

L.

	24	30	18
− 6	18	24	12
+ 6	30	36	24
÷ 3	8	10	6

M.

	8	10	15
x 4	32	40	60
− 8	0	2	7
+12	20	22	27

N.

	11	6	9
x 3	33	18	27
x 6	66	36	54
x 9	99	54	81

O.

	25	40	50
÷ 5	5	8	10
+15	40	55	65
− 6	19	34	44

P.

	6	10	8
x 5	30	50	40
+27	33	37	35
x 8	48	80	64

Q.

	21	49	56
−10	11	39	46
÷ 7	3	7	8
+16	37	65	72

R.

	72	48	56
÷ 8	9	6	7
− 9	63	39	47
x 0	0	0	0

S.

	45	27	54
x 2	90	54	108
÷ 9	5	3	6
−15	30	12	39

11-12 A diagnostic review of all basic facts of addition, subtraction; multiplication and division

Tasks: *In each of the exercises pupils carry out the required computations and record their results in the green squares provided.*

Purpose: To test recall of many of the basic facts of the arithmetic of whole numbers.

Unifying Ideas: Addition and subtraction; multiplication, and division.

The Lesson: Pupil pages 11 and 12—*Following Directions*—are designed to test pupil facility in the use of the basic facts of addition, subtraction, multiplication, and division.

Pupil page 11 calls for 131 responses: 33 in addition, 28 in subtraction, 42 in multiplication, and 28 in division. You can use 131 as a raw total score for the page, or you may prefer to interpret the four operations separately.

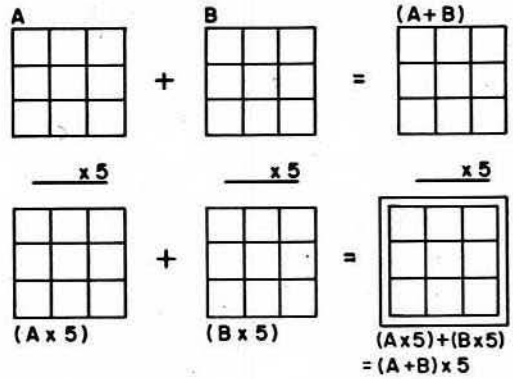
Pupils who score less than 90 per cent on this and the next page should make flash cards of the basic facts that they missed before going on in this work text. Those who score less than 70 per cent might better work in Level C than in Level D. And those who score less than 50 per cent should really work in Level B.

Rather than set a time limit, note the time each pupil takes to complete the page. (Normal working speeds and susceptibility to distractions vary greatly from pupil to pupil, and so you must rely on your judgment in deciding why particular individuals take more time to finish their work.)

As pupils complete the page, you may wish to give them practice sheet 11 to complete on their own. It is designed to extend the idea that Magic Squares behave much as numbers do. Examining the distributive principle for multiplication over addition, we notice:

10	(+)	7	(=)	17
× 5		× 5		× 5
50	(+)	35	(=)	85

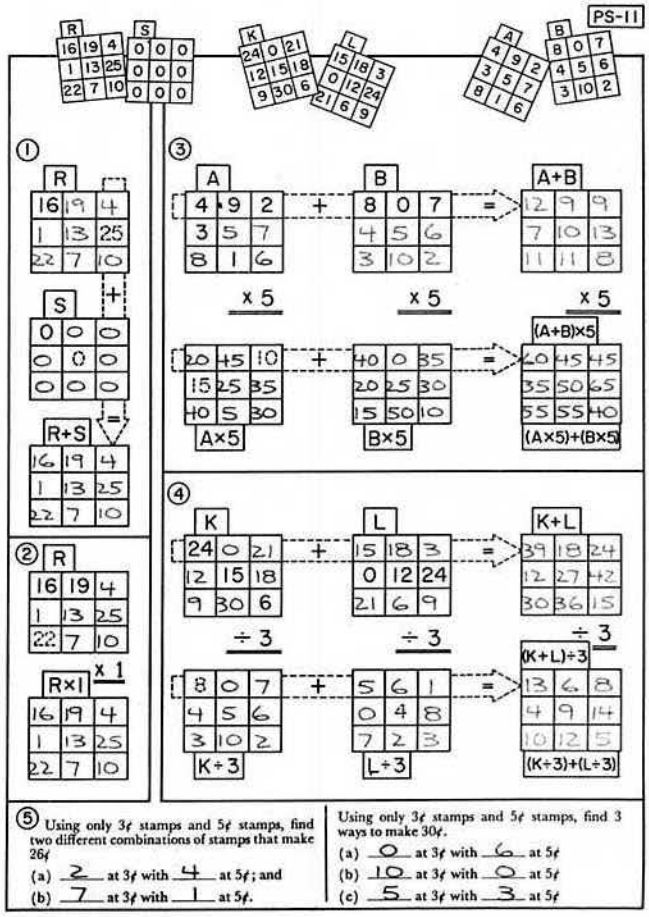
Example 3 of practice sheet 11 illustrates a similar principle for Magic Squares. We can get the same Magic Square by two different routes: (1) across the top row and down the right column; (2) down the left two columns and across the bottom row.



We note that: $(A+B)\times n = (A\times n)+(B\times n)$

Example 4 indicates that division is distributive over addition: $(K+L)\div n = (K\div n)+(L\div n)$

Example 1 illustrates that there is a Magic Square (all zeros) that, added to any Magic Square, gives that Magic Square as the sum. Example 2 illustrates that there is a number (one) such that any Magic Square multiplied by that number gives that Magic Square as the product.



A.

	12	24	6	30	18	36
÷6	2	4	1	5	3	6
−5	7	19	1	25	13	31
+9	21	33	15	39	27	45
÷3	4	8	2	10	6	12
x2	24	48	12	60	36	72

B.

	20	32	16	40	36
−3	17	29	13	37	33
÷2	10	16	8	20	18
x2	40	64	32	80	72
÷4	5	8	4	10	9
−16	4	16	0	24	20

C.

+	2	6	4	1	8	3	0	5	10	2	7	9	16
3	5	9	7	4	11	6	3	8	13	5	10	12	19
1	3	7	5	2	9	4	1	6	11	3	8	10	17
4	6	10	8	5	12	7	4	9	14	6	11	13	20
0	2	6	4	1	8	3	0	5	10	2	7	9	16
5	7	11	9	6	13	8	5	10	15	7	12	14	21
7	9	13	11	8	15	10	7	12	17	9	14	16	23
9	11	15	13	10	17	12	9	14	19	11	16	18	25
6	8	12	10	7	14	9	6	11	16	8	13	15	22
2	4	8	6	3	10	5	2	7	12	4	9	11	18
8	10	14	12	9	16	11	8	13	18	10	15	17	24

D.

X	2	6	4	1	8	3	0	5	10	2	7	9	16
3	6	18	12	3	24	9	0	15	30	6	21	27	48
1	2	6	4	1	8	3	0	5	10	2	7	9	16
4	8	24	16	4	32	12	0	20	40	8	28	36	64
0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	10	30	20	5	40	15	0	25	50	10	35	45	80
7	14	42	28	7	56	21	0	35	70	14	49	63	112
9	18	54	36	9	72	27	0	45	90	18	63	81	144
6	12	36	24	6	48	18	0	30	60	12	42	54	96
2	4	12	8	2	16	6	0	10	20	4	14	18	32
8	16	48	32	8	64	24	0	40	80	16	56	72	128

Notes on Pupil Page 12

Tasks: *In exercises A-D pupils carry out the required computations and record their results in the green spaces provided.*

Purpose: To test recall of the basic facts of the arithmetic of whole numbers.

Unifying Ideas: Addition and subtraction; multiplication and division.

The Lesson: Pupil page 12 is similar to pupil page 11 and can be used to take further inventory of pupil mastery of the basic facts. (The two large blocks include the 100 basic addition facts and the 100 basic multiplication facts.) Practice sheet 12 is simply an extension of the activity initiated by practice sheet 7.

You may wish to stimulate interest by urging pupils to look for shortcuts in completing this page. Go over several such shortcuts on the board. For example, in exercise A the first row calls for division by 6, and the fourth row for division by 3. A pupil might note that the fourth row can be completed by doubling the answers in the first row:

	12	24	6	30	18	36
÷6	2	4	1	5	3	6
÷3	4	8	2	10	6	12

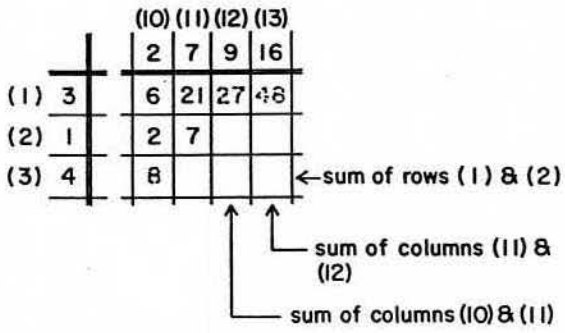
He might also note that the numbers heading the second and third columns are respectively double and half the one heading the first column, and that the number heading the fourth column is the sum of those in the second and third columns.

Once the first row in square C has been completed, each successive row can be derived from the one above it:

		2	6	4	1	
(1)	3	5	9	7	4	
(2)	1	3	7	5	2	2 less than row (1)
(3)	4	6	10	8	5	3 more than row (2)
(4)	0	2	6	4	1	4 less than row (3)
(5)	5					5 more than row (4)

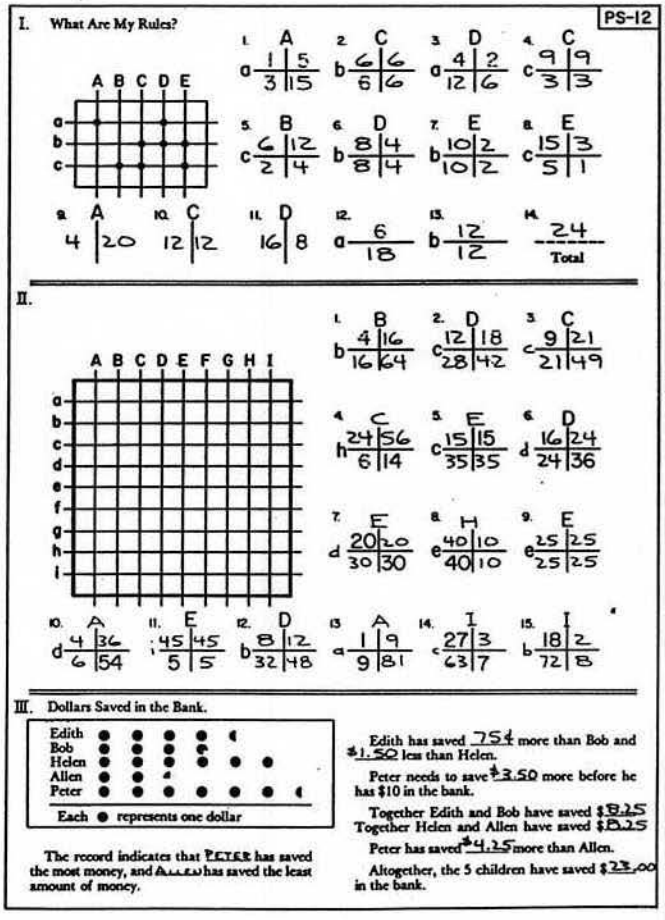
etc...

Similar shortcuts can be developed for the multiplication table in exercise D:



Pupils who notice such shortcuts probably do not need the practice they avoid. In fact the strategies noted above are precisely those used in the general multiplication algorithm:

X	10	7	17
10	100	70	170
3	30	21	51
13	130	91	221


PS-12
I. What Are My Rules?
II.
III. Dollars Saved in the Bank.
Edith
Bob
Helen
Allen
Peter
Each ● represents one dollar
Edith has saved 75¢ more than Bob and $1.50 less than Helen.
Peter needs to save $3.50 more before he has $10 in the bank.
Together Edith and Bob have saved $8.25
Together Helen and Allen have saved $8.25
Peter has saved $4.25 more than Allen.
Altogether, the 5 children have saved $23.00 in the bank.
The record indicates that PETER has saved the most money, and ALLEN has saved the least amount of money.

Ben and Ruth Harris are Newcomers

Mrs. Brown introduced two new students to her class at Roosevelt School.

"I would like you all to meet Ben Harris and his sister, Ruth Harris. They moved to town this summer.

"Ben tells me that they have lived in a part of the mountains where there are no schools. Their mother has taught them at home.

"Did your mother have a class in arithmetic for you?" Mrs. Brown asked Ben.

"Well, she had a longer name for it; she called it a class in the Arithmetic of Things and Numbers.

"We learned how to add and subtract things, and we learned how to multiply and divide things by numbers."

The students in the class were truly puzzled by what Ben said. It sounded familiar and yet somewhat strange at the same time. They were glad that Mrs. Brown asked more questions.

"Did you learn to add 2 and 6?"

"Two what and six what?" Ben asked. "You see, we only learned how to add and subtract things!"

"Can you explain what you mean?"

Ben Explains

"You see, we learned that

2 stars + 6 stars = 8 stars
2 feet + 6 inches = $2\frac{1}{2}$ feet or 30 in.
2 cows + 6 pigs = 8 animals

and we learned that there isn't a very sensible answer to the question:

2 miles + 6 quarts = ?

"So, you see, I need to know what things you are talking about."

"Tell us about subtraction," Mrs. Brown suggested. "How much is seven minus two — or, seven take away two?"

Ben scratched his head. "Again, I need to know your answers to 7 what? and 2 what?

"Mother taught us that

7 marbles − 3 marbles = 4 marbles
7 hours − 3 minutes = 6 hrs., 57 min.
7 yards − 3 feet = 6 yards

and she also taught us that there was no very sensible answer to:

7 cows − 3 chickens = ?

"Do you see why I need to know what things it is you are talking about?"

Mrs. Brown smiled. "What you say certainly makes sense."

As we continue telling you the story of the newcomers — Ben and Ruth — we will leave blanks. Please fill in the blanks so that the story will make sense to anyone who might be reading it.

You will need to learn the special ways that Mrs. Harris taught Ben and Ruth.

Mrs. Brown turned to Ruth. "How much is 4 multiplied by 3?"

"Ben and I would have to ask you — 4 what? Here are the kinds of things we learned:

4 inches x 3 = 1 foot
4¢ x 3 = 12¢
4 quarters x 3 = $3.00
4 days x 3 = 1 wk. and 5 days
4 eggs x 3 = 1 dozen eggs

"We learned there's no sensible answer to:

4 stars x 3 stars = ?

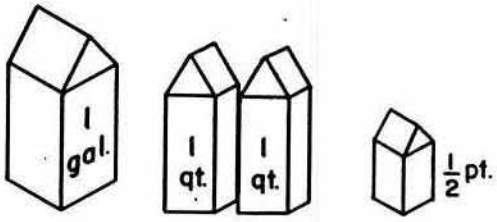
13-19 Basic operations of addition, subtraction, multiplication, and division, applied in the area of measurement

Tasks: Pupils read the introduction to the story of the Arithmetic of Things and Numbers *and supply the missing information necessary to complete the last two statements as true sentences.*

Purpose: To introduce pupils to the story of the *Arithmetic of Things and Numbers.*

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

The Lesson: Most applications of arithmetic are concerned with counting things—particularly things like apples or some chosen unit when making measurements. When we count and when we report the results of our counting we must realize the importance of knowing just what it is we are counting.



Here are some cartons of milk. Any questions about "how many" or "how much" must specify what we are talking about and what we wish to report. Otherwise, we must accept as correct any of the following responses:

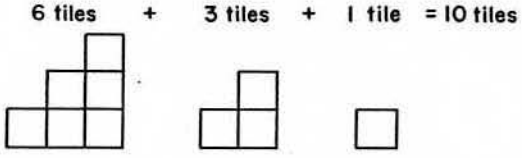
4 cartons
more than a gallon and a half of milk
25 glasses of milk
lots of cardboard
6½ quarts of milk

On pupil page 13 Ben and Ruth Harris are introduced. They in turn introduce the class to what they call the *Arithmetic of Things and Numbers.* This story continues through pupil page 23 and will be picked up again later in pupil pages 66 through 79.

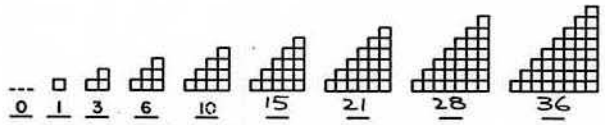
The authors feel very strongly that pupils make many mistakes because they think in terms of rules and marks on paper rather than about the real-world situations that these attempt to describe. Further we feel that many children think of arithmetic as a mysterious activity in which accepted results depend on following a specified routine of tricks. A central theme of Level D is to dispel this mystery and dependence on rules by studying many of the ways in which arithmetic is used to report events in the real world.

Practice sheet 13 is concerned with the relationship among a set of numbers generated by building larger and larger "stair-step" arrangements of tiles. We hope you will make cards of these arrangements of tiles so that pupils who have trouble can count to be sure.

You may wish to set these cards on the chalkrail and record above them the results of these additions or subtractions.



STAIR-STEP NUMBERS PS-13



0 1 3 6 10 15 21 28 36

The above numbers are called "Triangular Numbers."

Here is a way to see how the sequence grows:

Triangular numbers →	0	1	3	6	10	15	21	28	36	45	55	66	78	91
Differences →		1	2	3	4	5	6	7	8	9	10	11	12	13

Please use only triangular numbers to make true statements below:

30 = 28 + 1 + 1
31 = 28 + 3 + 0
32 = 28 + 3 + 1
33 = 15 + 15 + 3
40 = 36 + 3 + 1
41 = 21 + 10 + 10
42 = 36 + 6 + 0
43 = 36 + 6 + 1
44 = 28 + 10 + 6
50 = 28 + 21 + 1
51 = 45 + 6 + 0
52 = 45 + 6 + 1
53 = 28 + 15 + 10
54 = 45 + 6 + 3
60 = 45 + 15 + 0
61 = 55 + 6 + 0
62 = 55 + 6 + 1
63 = 45 + 15 + 3
64 = 55 + 6 + 3
65 = 55 + 10 + 0
70 = 66 + 3 + 1
71 = 55 + 10 + 6
72 = 66 + 6 + 0
73 = 66 + 6 + 1
74 = 45 + 28 + 1
75 = 66 + 6 + 3
76 = 66 + 10 + 0
83 = 55 + 28 + 0
84 = 78 + 6 + 0
85 = 78 + 6 + 1
86 = 66 + 10 + 10
90 = 78 + 6 + 6
95 = 91 + 3 + 1
96 = 78 + 15 + 3
99 = 78 + 21 + 0
100 = 78 + 21 + 1

Mary has saved $3.50; Alice has saved $1.45; and Janet has saved only 87¢.

Together they have saved $ 5.82

If each girl had saved twice as much as she did, together they would have saved $ 11.64

If Janet had saved as much as Alice did, the three girls would have $ 6.40

Mary saved $ 2.05 more than Alice saved and $ 2.63 more than Janet saved.

Janet saved 58 ¢ more than Alice saved.

Their goal was to save a total of $10.00. They are $ 4.18 short of their goal.

Janet plans to save half of the amount the girls set as their goal, or $ 5.00

"In division we learned this kind of idea:

1 doz. eggs ÷ 3 = 4 eggs
1 gallon ÷ 4 = 1 quart
1 week and 3 days ÷ 2 = 5 days
12 chocolates ÷ 3 = 4 chocolates
20¢ ÷ 4 = 5 ¢
1 hour ÷ 3 = 20 minutes

"Mother used to put the general idea in this way:

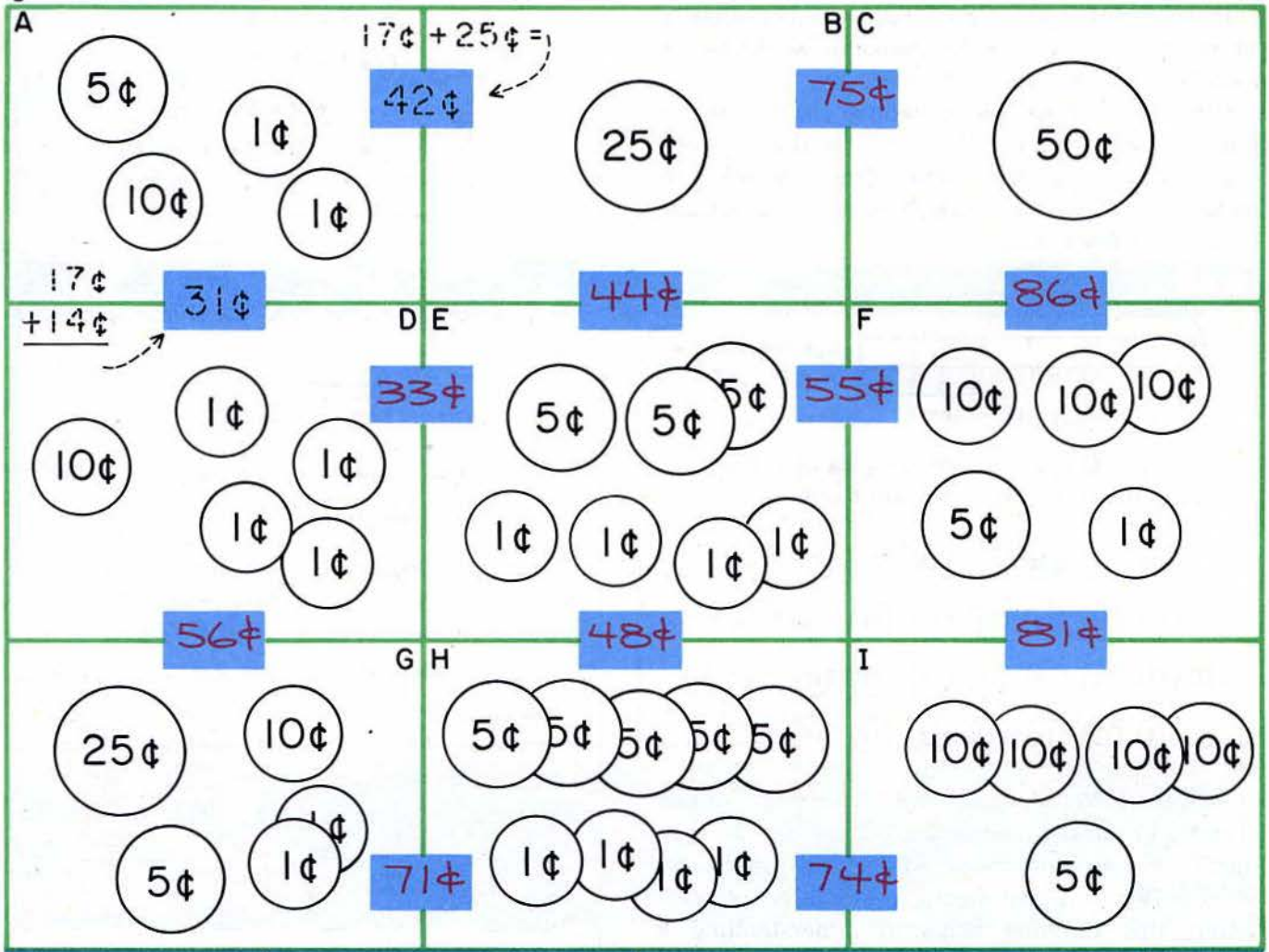
Things + Things = Things
Things − Things = Things
Things x a number = Things
Things ÷ a number = Things

"And she said it is most important to think about what you are saying so you can be sure you are talking sense — and not nonsense."

Mrs. Brown visited Mrs. Harris and talked about her "Arithmetic of Things and Numbers." Mrs. Brown asked to see some of the work that Ben and Ruth had done.

The pages that follow are samples Mrs. Harris gave to Mrs. Brown. Please fill in all the blanks you can.

I



II

a	b	c	d	e	f
29¢ + 32¢ = 61¢	78¢ − 32¢ = 46¢	61¢ + 19¢ = 80¢	23 miles − 8 miles = 15 miles	46 pounds + 46 pounds = 92 pounds	54 minutes − 17 minutes = 37 minutes

Notes on Pupil Page 14

Tasks: *Pupils continue reading the story of the* Arithmetic of Things and Numbers. *In exercise I pupils use the clues provided to record in the blue spaces the sums of the values of collections of coins displayed in adjacent pairs of rectangular regions. In exercise II pupils compute and record sums and differences.*

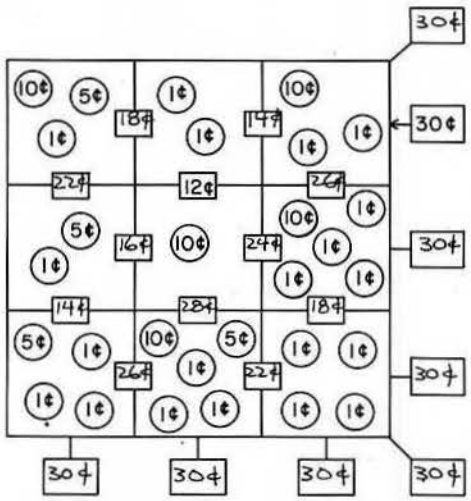
Purpose: To perfect computational skills with various units of measure by exploring the *Arithmetic of Things and Numbers.*

Unifying Ideas: Addition and subtraction; measurement.

The Lesson: Ruth's conclusions at the top of pupil page 14 have a familiar look. We could substitute "Magic Squares" for the word "Things" and each expression could indicate a property of an operation on Magic Squares. In fact we can think of our study of Magic Squares as part of the *Arithmetic of Things and Numbers.*

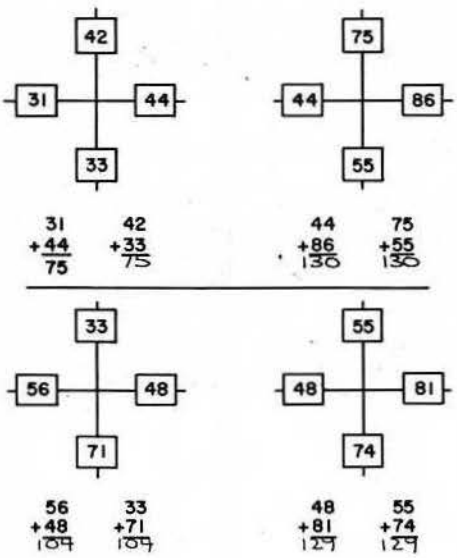
Practice sheet 14 includes some of the tables of measurement needed for the work ahead.

Here is a variation of exercise I of pupil page 14. It provides a place to record the total amounts in each row, column, and diagonal.

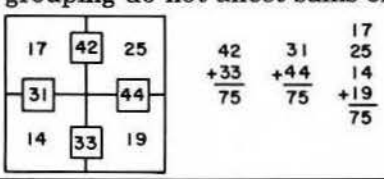


Of course, the arrangement of coins is based on a Magic Square.

In the example on pupil page 14 and the one above there is a relationship worth noting. Consider each cluster of four sums. In each case the sum across, the sum down, and the sum of the numbers in the four boxes is the same.



Order and grouping do not affect sums of numbers.



PS-14

1 yard = 3 feet = 36 inches. 1 quarter = 5 nickels = 25 pennies.

1 gallon = 4 quarts = 8 pints. 1 pound = 16 ounces.

1 dozen eggs = 2 half-dozen eggs = 12 eggs. 1 week = 7 days.

1 dollar = 2 half-dollars = 4 quarters = 10 dimes = 20 nickels = 100 pennies.

① yards, feet, and inches

1 yd. = 3 ft. = 36 in.
2 yd. = 6 ft. = 72 in.
3 yd. = 9 ft. = 108 in.
5 yd. = 15 ft. = 180 in.

② quarters (q), nickels (n), and pennies (p)

1 q. = 5 n. = 25 p.
3 q. = 15 n. = 75 p.
4 q. = 20 n. = 100 p.
7 q. = 35 n. = 175 p.

③ gallons, quarts, and pints

1 gal. = 4 qt. = 8 pt.
2 gal. = 8 qt. = 16 pt.
4 gal. = 16 qt. = 32 pt.
7 gal. = 28 qt. = 56 pt.

④ pounds and ounces

1 lb. = 16 oz.
2 lb. = 32 oz.
4 lb. = 64 oz.
6 lb. = 96 oz.

⑤ dozens of eggs, half-dozens, and eggs

1 doz. = 2 half-doz. = 12 eggs
2 doz. = 4 half-doz. = 24 eggs
4 doz. = 8 half-doz. = 48 eggs
6 doz. = 12 half-doz. = 72 eggs
10 doz. = 20 half-doz. = 120 eggs

⑥ weeks and days

1 wk. = 7 da.
3 wk. = 21 da.
4 wk. = 28 da.
7 wk. = 49 da.
14 wk. = 98 da.

⑦ dollars, half-dollars, quarters, dimes, nickels, and pennies

1 dol. = 2 half-dol. = 4 q. = 10 d. = 20 n. = 100 p.
2 dol. = 4 half-dol. = 8 q. = 20 d. = 40 n. = 200 p.
3 dol. = 6 half-dol. = 12 q. = 30 d. = 60 n. = 300 p.
5 dol. = 10 half dol. = 20 q. = 50 d = 100 n. = 500 p.
10 dol. = 20 half dol. = 40 q. = 100 d. = 200 n. = 1000 p.
15 dol. = 30 half dol. = 60 q. = 150 d. = 300 n. = 1500 p.

Please fill in the blanks so that the sentences make sense.

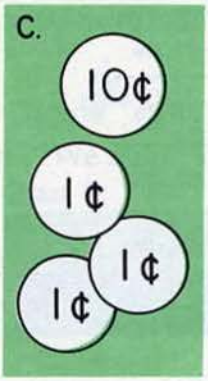
A.

4 nickels + 3 dimes = 50 ¢
1 hour − 10 minutes = 50 minutes
1 week + 6 days = 13 days
2 feet + 8 inches = 32 inches
1 quarter + 2 quarters = 75 ¢
1 yard − 12 inches = 2 feet
38 minutes + 17 minutes = 55 min.
17 pennies + 9 nickels = 62 ¢

B.

13 days + 8 days = 3 weeks
1 gallon − 1 quart = 3 quarts
2 weeks − 9 days = 5 days
$1.50 − 75¢ = 75 ¢
7 nickels + 9 nickels = 8 dimes
80 minutes + 40 minutes = 2 hrs.
2 pounds − 20 ounces = 12 ounces
8 yards − 19 feet = 5 feet

C.



. . . + a quarter = 38 ¢
. . . + 47¢ = 60 ¢
. . . − 4¢ = 9 ¢
. . . + 7 nickels = 48 ¢
. . . + 3 quarters = 88 ¢
. . . + 12 dimes = $1.33

. . . x 2 = 26 ¢
. . . x 3 = 39 ¢
. . . x 5 = 65 ¢
. . . x 8 = $1.04
. . . x 4 = 52 ¢
. . . x 6 = 78 ¢

Please fill in the blanks so that the sentences make sense.

D.

3 days x 2 = 6 days
13 minutes x 2 = 26 minutes
13 minutes x 3 = 39 minutes
14 eggs ÷ 2 = 7 eggs
1 foot ÷ 2 = 6 inches
1 nickel x 4 = 20 ¢
6 inches x 4 = 2 feet
30 minutes x 4 = 2 hours

E.

1 hour ÷ 2 = 30 minutes
1 hour ÷ 3 = 20 minutes
1 hour ÷ 4 = 15 minutes
1 gallon ÷ 2 = 2 quarts
2 feet ÷ 3 = 8 inches
28¢ ÷ 2 = 14 ¢
12 chocolates ÷ 3 = 4 chocolates
18 miles ÷ 3 = 6 miles

F.

1 week	$1.17	2 feet	90 minutes	5 yards
−3 days	+ 1.35	− 18 inches	−35 minutes	+7 feet
4 days	$2.52	6 INCHES	55 MIN.	22 feet

G.

15 miles	25¢	6 inches	8 ounces	30 min.
x 2	3) 75¢	x 4	4) 2 pounds	x 6
30 miles		2 feet		3 hrs.

Tasks: *Pupils perform the necessary computations and record their results in the blanks provided so that true statements are obtained.*

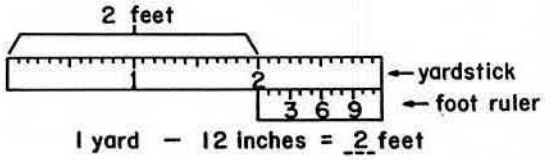
Purpose: To perfect computational skills with various units of measure by exploring the *Arithmetic of Things and Numbers.*

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; measurement.

The Lesson: Pupil page 15 will help emphasize the importance of reading carefully and giving the response that is called for. Questions in this form may be new to some pupils. If they notice only the numbers—not the units—they may write "4 nickels +3 dimes = 7¢."

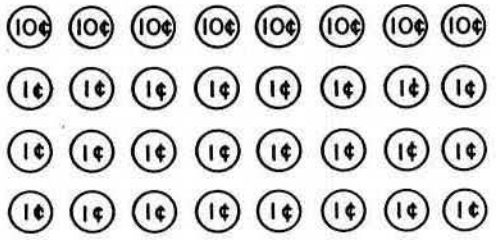
If the question did not call for a number of cents as an answer, a reasonable response would be "4 nickels + 3 dimes = 7 coins."

Others may form the statement "4 nickels + 3 dimes = 35¢," thinking that because the sentence begins with a number of nickels the 3 also refers to nickels, and 7 nickels is 35¢. It may help to act out or sketch a few examples.



1 yard − 12 inches = 2 feet

In exercise C if we act out the idea of 1 dime and 3 pennies multiplied by 8 we will have:



How much? Well, 8 dimes and 24 pennies. We can exchange 20 of the pennies for 2 dimes and have 10 dimes and 4 pennies—or $1.04. This is clearly sensible for thinking about dimes and pennies. Later, this thinking helps in understanding a standard algorithm for multiplication:

$$\begin{array}{r} 13 \\ \times\ 8 \\ \hline 104 \end{array}$$

Each of the seven groups of examples can be used as a model for additional practice.

Practice sheet 15 indicates a particular set of numbers that we call "bowling pin" numbers. Notice that regardless of the difference in the sketch, this is the same collection of numbers considered on practice sheet 13—the triangular numbers.

After noting the way in which the sketches are made, pupils should not be surprised by the following results.

$0 = 0$

$1 = 0 + 1$

$3 = 0 + 1 + 2$

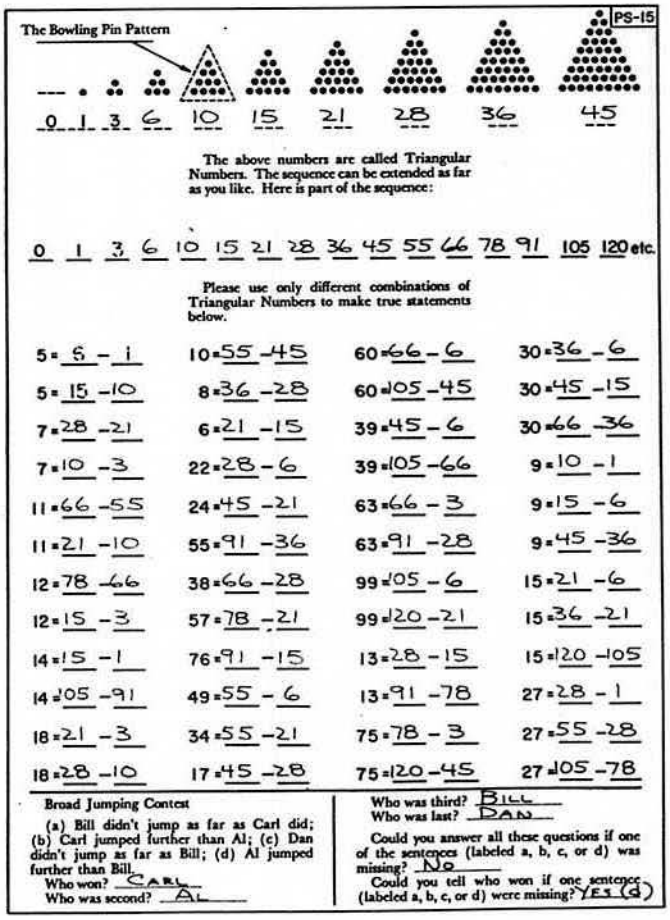
$6 = 0 + 1 + 2 + 3$

$10 = 0 + 1 + 2 + 3 + 4$

$15 = 0 + 1 + 2 + 3 + 4 + 5$

$21 = 0 + 1 + 2 + 3 + 4 + 5 + 6$

etc.


PS-15

The Bowling Pin Pattern

0 1 3 6 10 15 21 28 36 45

The above numbers are called Triangular Numbers. The sequence can be extended as far as you like. Here is part of the sequence:

0 1 3 6 10 15 21 28 36 45 55 66 78 91 105 120 etc.

Please use only different combinations of Triangular Numbers to make true statements below.

5 = 6 − 1	10 = 55 − 45	60 = 66 − 6	30 = 36 − 6
5 = 15 − 10	8 = 36 − 28	60 = 105 − 45	30 = 45 − 15
7 = 28 − 21	6 = 21 − 15	39 = 45 − 6	30 = 66 − 36
7 = 10 − 3	22 = 28 − 6	39 = 105 − 66	9 = 10 − 1
11 = 66 − 55	24 = 45 − 21	63 = 66 − 3	9 = 15 − 6
11 = 21 − 10	55 = 91 − 36	63 = 91 − 28	9 = 45 − 36
12 = 78 − 66	38 = 66 − 28	99 = 105 − 6	15 = 21 − 6
12 = 15 − 3	57 = 78 − 21	99 = 120 − 21	15 = 36 − 21
14 = 15 − 1	76 = 91 − 15	13 = 28 − 15	15 = 120 − 105
14 = 105 − 91	49 = 55 − 6	13 = 91 − 78	27 = 28 − 1
18 = 21 − 3	34 = 55 − 21	75 = 78 − 3	27 = 55 − 28
18 = 28 − 10	17 = 45 − 28	75 = 120 − 45	27 = 105 − 78

Broad Jumping Contest

(a) Bill didn't jump as far as Carl did; (b) Carl jumped further than Al; (c) Dan didn't jump as far as Bill; (d) Al jumped further than Bill.

Who won? CARL

Who was second? AL

Who was third? BILL

Who was last? DAN

Could you answer all these questions if one of the sentences (labeled a, b, c, or d) was missing? NO

Could you tell who won if one sentence (labeled a, b, c, or d) were missing? YES (d)

Ruth explained how she and Ben used to make charts for their mother.

"Sometimes Mother would write very easy questions for us to answer. Here is a sample:

How many quarts in 3 gallons ?

"Or she might ask us to complete a sentence like the following:

3 gal. = 12 quarts

"That's so easy that Ben and I made a list of answers to many sentences of the form:

▲ gal. = ■ qts.

"We might start the list this way:

Plan A.

gal.	0	1	2	3	4	5	6	etc.
qts.	0	4	8	12	16	20	24	

"You can fill in the blanks. We would put an arrow to show Mother where she could look for the numbers for the easy sentence she had asked us to complete.

"Sometimes we used a different plan for the top line in our list. As you fill in the blanks, you may see why we chose this plan.

Plan B.

g	1	2	3	5	8	13	etc.
q	4	8	12	20	32	52	

"Or, we might decide on a different plan. I'll start it. Please finish the top line and fill in the blanks."

Plan C.

g	1	2	4	8	16	32	64
q	4	8	16	32	64	128	256

"Here's another plan."

Plan D.

g	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$
q	2	4	6	8	10	12	14

"Show us another example," Bill suggested.

"All right," Ruth said. "I think you would agree that this sentence:

2 nickels = 10 pennies

is much too easy to worry about. So Ben and I would make up lists like these. I am sure you can complete them . . . and, if you like, extend them."

▲ nickels = ■ pennies

Plan E.

nickels	0	1	2	3	4	5
pennies	0	5	10	15	20	25

Plan F.

n	2	4	6	10	16	20
p	10	20	30	50	80	100

Plan G.

n	1	2	4	8	16	32
p	5	10	20	40	80	160

"Here's another question that's too easy:

How many days in 3 weeks ?

"So, we made several lists for Mother. We thought of a different kind of sentence:

▲ weeks = ■ days

Plan H.

weeks	0	1	2	3	4	5
days	0	7	14	21	28	35

Plan I.

w	3	6	12	24	48	96
d	21	42	84	168	336	672

"You may want to extend these lists or make some with plans of your own."

Tasks: Pupils use the clues provided to determine and apply the conversion patterns necessary to complete each of the tables.

Purpose: To perfect computational skills with various units of measure by exploring the *Arithmetic of Things and Numbers.*

Unifying Ideas: Numeration; multiplication and division; measurement.

The Lesson: One way to look at applications of arithmetic might be called the "customer approach." A customer may need to know that 3 gallons are equivalent to 12 quarts, or that 2 nickels are equivalent to 10 pennies. Another way to look at applications might be called the "clerk's approach." A clerk needs to know the number of quarts in 1 gallon, 2 gallons, 3 gallons, etc. He needs to know that 1 nickel is equivalent to 5¢, 2 nickels are equivalent to 10¢, etc.

If adult tickets are 75¢ each and children's tickets are 35¢ each, a customer who wishes to purchase one of each type of ticket would need to know that 75¢+35¢=$1.10. But the ticket seller must know much more. He has to know all the entries in an extension of the chart below or know how to find the entries.

children @ 35¢ \ adults @ 75¢	0	1	2	3
0		75¢	$1.50	$2.25
1	35¢	$1.10	$1.85	$2.60
2	70¢	$1.45	$2.20	$2.95
3	$1.05	$1.80	$2.55	$3.30

In fact, clerks may actually make such charts to avoid having to compute the amount of each sale as it is made. Banks make comparable charts of amounts of interest at different rates and for different lengths of time.

Almost any "word problem" out of an old arithmetic text or from a standardized achievement test is an example of customer arithmetic. Each such item leads to a much more interesting situation—the clerk's arithmetic, of which the given problem is but a small part. For example:

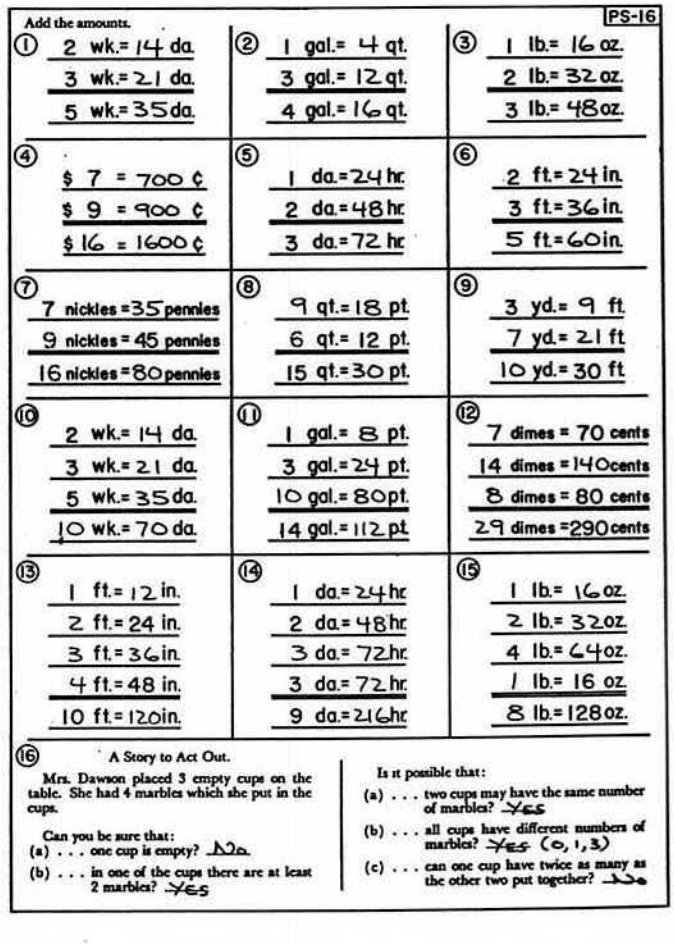
Soup is 13¢ a can or 2 for a quarter.
How much do 3 cans cost?

is a question a customer considers. One example of what the checkout clerk needs to know is:

Cans	1	2	3	4	5	6	7	
Cost	13¢	25¢	38¢	50¢	63¢	75¢	88¢	etc.

The examples on practice sheet 16 can be used to suggest similar charts. For example:

+	0 wk	1 wk	2 wk	3 wk	
0 wk	0 da	7 da	14 da	21 da	
1 wk	7 da	14 da	21 da	28 da	etc.
2 wk	14 da	21 da	28 da	35 da	
3 wk	21 da	28 da	35 da	42 da	
		etc.			


PS-16
Add the amounts.
① 2 wk.= 14 da. 3 wk.= 21 da. 5 wk.= 35 da.
② 1 gal.= 4 qt. 3 gal.= 12 qt. 4 gal.= 16 qt.
③ 1 lb.= 16 oz. 2 lb.= 32 oz. 3 lb.= 48 oz.
④ $ 7 = 700 ¢ $ 9 = 900 ¢ $ 16 = 1600 ¢
⑤ 1 da.= 24 hr. 2 da.= 48 hr. 3 da.= 72 hr.
⑥ 2 ft.= 24 in. 3 ft.= 36 in. 5 ft.= 60 in.
⑦ 7 nickles = 35 pennies 9 nickles = 45 pennies 16 nickles = 80 pennies
⑧ 9 qt.= 18 pt. 6 qt.= 12 pt. 15 qt.= 30 pt.
⑨ 3 yd.= 9 ft. 7 yd.= 21 ft. 10 yd.= 30 ft.
⑩ 2 wk.= 14 da. 3 wk.= 21 da. 5 wk.= 35 da. 10 wk.= 70 da.
⑪ 1 gal.= 8 pt. 3 gal.= 24 pt. 10 gal.= 80 pt. 14 gal.= 112 pt.
⑫ 7 dimes = 70 cents 14 dimes = 140 cents 8 dimes = 80 cents 29 dimes = 290 cents
⑬ 1 ft.= 12 in. 2 ft.= 24 in. 3 ft.= 36 in. 4 ft.= 48 in. 10 ft.= 120 in.
⑭ 1 da.= 24 hr. 2 da.= 48 hr. 3 da.= 72 hr. 3 da.= 72 hr. 9 da.= 216 hr.
⑮ 1 lb.= 16 oz. 2 lb.= 32 oz. 4 lb.= 64 oz. 1 lb.= 16 oz. 8 lb.= 128 oz.
⑯ A Story to Act Out.
Mrs. Dawson placed 3 empty cups on the table. She had 4 marbles which she put in the cups.
Can you be sure that:
(a) . . . one cup is empty? No.
(b) . . . in one of the cups there are at least 2 marbles? YES
Is it possible that:
(a) . . . two cups may have the same number of marbles? YES
(b) . . . all cups have different numbers of marbles? YES (0, 1, 3)
(c) . . . can one cup have twice as many as the other two put together? No.

The children in Mrs. Brown's class wanted more examples; so Ruth made up several:

Mother's question:

If cakes are packed a half-dozen in a box, how many cakes are there in 3 boxes?

Our sentence:

▲ half-dozens = ■ cakes

Plan J.

half-doz.	0	1	2	3	4	5
cakes	0	6	12	18	24	30

Plan K.

half-doz.	5	10	15	20	25	30
cakes	30	60	90	120	150	180

Plan L.

half-doz.	4	2	6	8	14	7
cakes	24	12	36	48	84	42

Jumping around (with some planning).

half-doz.	6	10	16	20	8	28
cakes	36	60	96	120	48	168

Mother's sentence: 1 foot = ____ inches

Our sentence: ▲ feet = ■ in.

Plan M.

feet	0	1	2	3	4	5
inches	0	12	24	36	48	60

Plan N.

ft.	2	4	6	10	16	26
in.	24	48	72	120	192	312

Jumping around (with some planning).

ft.	2	10	12	3	20	23
in.	24	120	144	36	240	276

Mother's sentences:

1 gallon = ____ half-gallon
1 gallon = ____ quarts
1 gallon = ____ pints
1 gallon = ____ half-pints

Our list shows the completions for these sentences — and many more sentences.

Plan O.

gallons	1	2	3	4	5	6
half-gal.	2	4	6	8	10	12
quarts	4	8	12	16	20	24
pints	8	16	24	32	40	48
half-pints	16	32	48	64	80	96

Plan P.

gal.	4	3	7	10	17	27
$\frac{1}{2}$ gal.	8	6	14	20	34	54
qts.	16	12	28	40	68	108
pts.	32	24	56	80	136	216
$\frac{1}{2}$ pts.	64	48	112	160	272	432

Mother's sentence: 2 hours = ____ minutes

Too easy! So, we make tables like the following:

Plan Q.

hours	1	2	3	4	5
minutes	60	120	180	240	300

Plan R.

hr.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$
min.	15	30	45	60	75	90

Plan S.

hr.	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$
min.	20	40	60	80	100	120	140

Tasks: Pupils use the clues provided and complete each of the tables.

Purpose: To perfect computational skills with various units of measure by exploring the *Arithmetic of Things and Numbers.*

Unifying Ideas: Numeration; multiplication and division; measurement.

The Lesson: Another way to expand on the traditional word problem is to leave out some of the information:

Bill had ____ marbles and found ____ more. Now he has 7 marbles.

We haven't enough information to know how many he had or found, but we can easily list all possible pairs of numbers that could be used:

Had	0	1	2	3	4	5	6	7
Found	7	6	5	4	3	2	1	0

Oranges are 3¢ each or 2 for 5¢, and apples are 4¢ each or 3 for 10¢. Bill bought 6 pieces of fruit.

We can list all possible combinations and note total costs:

apples	0	1	2	3	4	5	6
oranges	6	5	4	3	2	1	0
Cost of apples	0	4	8	10	14	18	20
Cost of oranges	15	13	10	8	5	3	0
Total cost	15	17	18	18	19	21	20

These results suggest many questions that are far from obvious without the chart: What did he buy if he spent 17¢? 19¢? 15¢? Etc. What amount of money could be spent in two different ways? How many different purchases could be paid for with exactly six coins? (18¢: 3 nickels and 3 pennies; 15¢: 1 dime and 5 pennies; etc.)

All "insufficient information" problems require much more than simple recall. They require organization of the results of recall and interpretation of those results.

Practice sheet 17 deals with a familiar collection of numbers, the square numbers—last seen as a diagonal in the multiplication table on pupil page 10. Pupils are to find a combination of four square numbers whose sum is a specified number. (You may be interested to know that it has been proved that every whole number can be expressed as the sum of four square numbers—and some in several different ways, as you will note on practice sheet 19.)

Square numbers can be generated in another way. The sum of each successive pair of triangular numbers is a square number:

Triangular No's: 0 1 3 6 10

1 4 9 16

and in another way

1 = 1

4 = 1 + 3

9 = 1 + 3 + 5

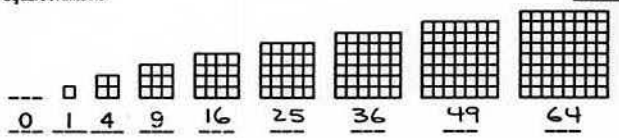
16 = 1 + 3 + 5 + 7

25 = 1 + 3 + 5 + 7 + 9

36 = 1 + 3 + 5 + 7 + 9 + 11

49 = 1 + 3 + 5 + 7 + 9 + 11 + 13

Square Numbers PS-17



0 1 4 9 16 25 36 49 64

Here is another way to see how this sequence of numbers grows:

Square numbers: 0 1 4 9 16 25 36 49 64 81 100 121 144 etc.

Differences: 1 3 5 7 9 11 13 15 17 19 21 23

Please use only square numbers to make true statements below:

7 = 4 + 1 + 1 + 1	60 = 49 + 9 + 1 + 1	89 = 81 + 4 + 4 + 0
8 = 4 + 4 + 0 + 0	61 = 36 + 25 + 0 + 0	90 = 81 + 9 + 0 + 0
9 = 9 + 0 + 0 + 0	62 = 49 + 9 + 4 + 0	91 = 81 + 9 + 1 + 0
13 = 9 + 4 + 0 + 0	71 = 36 + 25 + 9 + 1	92 = 81 + 9 + 1 + 1
22 = 16 + 4 + 1 + 1	72 = 64 + 4 + 4 + 0	93 = 81 + 4 + 4 + 4
23 = 9 + 9 + 4 + 1	76 = 64 + 4 + 4 + 4	94 = 81 + 9 + 4 + 0
24 = 16 + 4 + 4 + 0	77 = 64 + 9 + 4 + 0	95 = 81 + 9 + 4 + 1
31 = 25 + 4 + 1 + 1	79 = 49 + 25 + 4 + 1	96 = 64 + 16 + 16 + 0
32 = 16 + 16 + 0 + 0	80 = 64 + 16 + 0 + 0	97 = 81 + 16 + 0 + 0
45 = 36 + 9 + 0 + 0	85 = 81 + 4 + 0 + 0	98 = 49 + 49 + 0 + 0
56 = 36 + 16 + 4 + 0	87 = 81 + 4 + 1 + 1	99 = 49 + 25 + 25 + 0
59 = 49 + 9 + 1 + 0	88 = 64 + 16 + 4 + 4	100 = 64 + 36 + 0 + 0

Because of our system of place value, some digits are worth more than others. In each of the examples below, draw a loop around the digit that is worth most:

(a) (1) (c) (4) 7 (e) (2) 4 8

(b) (9) (d) (3) 7 (f) (9) 6 7

Instead of buying a suit for $19.65, Harry bought a jacket for $11.95 and a pair of slacks for $7.80.

Harry spent $ 19.75 That is MORE (more or less) than the suit cost. The difference is 10¢

Two suits would have cost $ 39.30 Two jackets would cost $ 23.10

"Of course," Ruth explained, "Mother sometimes gave us problems that were more interesting. Here is one:

1 nickel + 3 cents = _8_ cents

"The answer is easy enough, but we wanted to give more than just an easy answer. So, we made a chart in the form of:

____ nickels + ____ cents = ... cents

Plan A.

+	0¢	1¢	2¢	3¢	4¢	5¢
0 nickels	0¢	1¢	2¢	3¢	4¢	5¢
1 nickel	5¢	6¢	7¢	8¢	9¢	10¢
2 n	10¢	11¢	12¢	13¢	14¢	15¢
3 n	15¢	16¢	17¢	18¢	19¢	20¢

"Please fill in the rest of the chart."

"You notice," Ruth said, "that our chart includes the answer to Mother's problem — in the heavy black box.

"Here is another plan we might use for a chart."

Plan B.

+	2¢	4¢	6¢	8¢	10¢	12¢
2n	12¢	14¢	16¢	18¢	20¢	22¢
4n	22¢	24¢	26¢	28¢	30¢	32¢
6n	32¢	34¢	36¢	38¢	40¢	42¢
8n	42¢	44¢	46¢	48¢	50¢	52¢

Plan C.

+	3¢	7¢	11¢	15¢	19¢	23¢
1n	8¢	12¢	16¢	20¢	24¢	28¢
3n	18¢	22¢	26¢	30¢	34¢	38¢
5n	28¢	32¢	36¢	40¢	44¢	48¢
7n	38¢	42¢	46¢	50¢	54¢	58¢
9n	48¢	52¢	56¢	60¢	64¢	68¢

"Here is another one of mother's problems:

2 gals. + 3 qts. = _11_ qts.

"We would make a chart in the form of:

____ gals. + ____ qts. = ... qts.

"To save space, we would use g for gallons and q for quarts.

Plan D.

+	0q	1q	2q	3q	4q
0g	0q	1q	2q	3q	4q
1g	4q	5q	6q	7q	8q
2g	8q	9q	10q	11q	12q
3g	12q	13q	14q	15q	16q
4g	16q	17q	18q	19q	20q

"The answer to Mother's problem is in the heavy black-line box."

Jumping around

+	0q	4q	7q	10q	13q
5g	20q	24q	27q	30q	33q
10g	40q	44q	47q	50q	53q
6g	24q	28q	31q	34q	37q
12g	48q	52q	55q	58q	61q
7g	28q	32q	35q	38q	41q
14g	56q	60q	63q	66q	69q
8g	32q	36q	39q	42q	45q

"You might like to extend some of the charts — in both directions — down and to the right.

"Can you imagine a chart we might have made for this problem?"

3 dimes + 2 nickels = ____ cents

Notes on Pupil Page 18

Tasks: Pupils use the clues provided to carry out the computations to complete each table.

Purpose: To perfect computational skills with various units of measure by exploring the *Arithmetic of Things and Numbers.*

Unifying Ideas: Numeration; addition and subtraction; measurement.

The Lesson: We might consider each example on this page as transforming a "customer arithmetic" problem into a "clerk arithmetic" problem. Putting it another way, we are making an "insufficient information" problem out of a traditional (or almost traditional) word problem. The problem as shown is easily completed:

1 nickel + 3 cents = 8 cents

But Ben and Ruth look at the problem as if it read:

____ nickels + ____ cents = ____ cents

Then they set out to find some of the many triples of numbers that will make it a true statement. Notice that many plans are possible:

Plan A is to consider all possible combinations;

Plan B is to consider all possible combinations in which nonzero even numbers of nickels and pennies are employed;

Plan C is to consider odd numbers of nickels, starting with 1, and odd numbers of pennies, starting with 3 and skipping every other one.

These and many similar plans could never be completed since the chart can be expanded indefinitely. Here are three that can be completed:

Consider all combinations with totals of more than 9¢ and less than 21¢;

Consider all combinations involving exactly 8 coins;

Consider all combinations involving exactly 10 coins.

The question at the bottom of pupil page 18 suggests that we change the sentence:

3 dimes + 2 nickels = ____ cents

to the sentence:

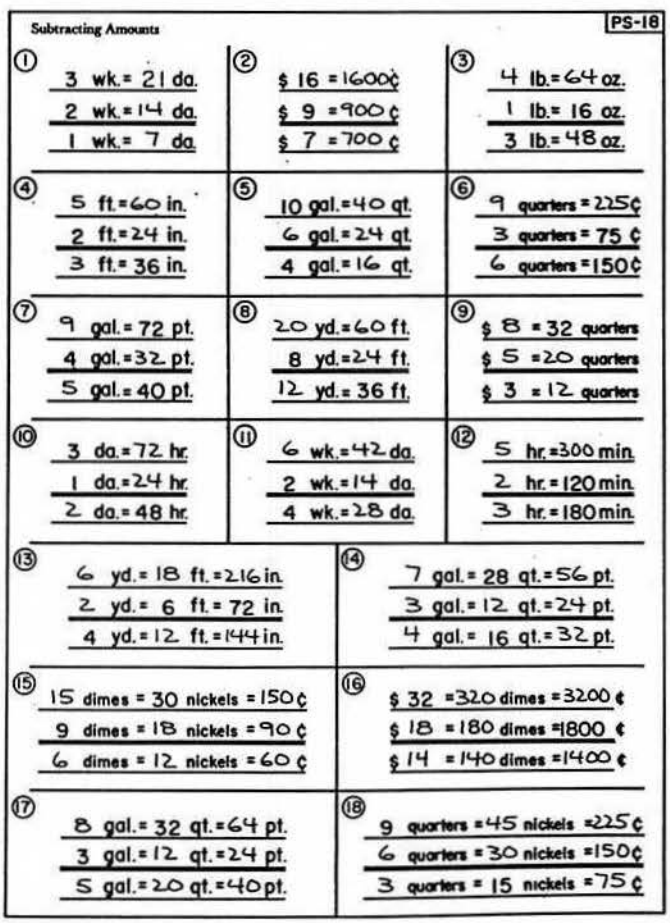
____ dimes + ____ nickels = ____ cents

A chart with which to begin to consider all combinations might look something like this:

	0n	1n	2n
0 d	0¢	5¢	10¢
1 d	10¢	15¢	20¢
2 d	20¢	25¢	30¢

Practice sheet 18 extends the activity of practice sheet 16 to include the idea of subtracting one amount from another. Notice that for subtraction or addition problems in this form, any two entries not on the same line give enough information to complete the example. This is true no matter how long the lines are:

1 gal. =	2 half gals. =	4 qts. =	8 pts. =	16 half pts.
+4 gals. =	8 half gals. =	16 qts. =	32 pts. =	64 half pts.
5 gals. =	10 half gals. =	20 qts. =	40 pts. =	80 half pts.



Subtracting Amounts — PS-18

① 3 wk. = 21 da.; 2 wk. = 14 da.; 1 wk. = 7 da.

② $16 = 1600¢; $9 = 900¢; $7 = 700¢

③ 4 lb. = 64 oz.; 1 lb. = 16 oz.; 3 lb. = 48 oz.

④ 5 ft. = 60 in.; 2 ft. = 24 in.; 3 ft. = 36 in.

⑤ 10 gal. = 40 qt.; 6 gal. = 24 qt.; 4 gal. = 16 qt.

⑥ 9 quarters = 225¢; 3 quarters = 75¢; 6 quarters = 150¢

⑦ 9 gal. = 72 pt.; 4 gal. = 32 pt.; 5 gal. = 40 pt.

⑧ 20 yd. = 60 ft.; 8 yd. = 24 ft.; 12 yd. = 36 ft.

⑨ $8 = 32 quarters; $5 = 20 quarters; $3 = 12 quarters

⑩ 3 da. = 72 hr.; 1 da. = 24 hr.; 2 da. = 48 hr.

⑪ 6 wk. = 42 da.; 2 wk. = 14 da.; 4 wk. = 28 da.

⑫ 5 hr. = 300 min.; 2 hr. = 120 min.; 3 hr. = 180 min.

⑬ 6 yd. = 18 ft. = 216 in.; 2 yd. = 6 ft. = 72 in.; 4 yd. = 12 ft. = 144 in.

⑭ 7 gal. = 28 qt. = 56 pt.; 3 gal. = 12 qt. = 24 pt.; 4 gal. = 16 qt. = 32 pt.

⑮ 15 dimes = 30 nickels = 150¢; 9 dimes = 18 nickels = 90¢; 6 dimes = 12 nickels = 60¢

⑯ $32 = 320 dimes = 3200¢; $18 = 180 dimes = 1800¢; $14 = 140 dimes = 1400¢

⑰ 8 gal. = 32 qt. = 64 pt.; 3 gal. = 12 qt. = 24 pt.; 5 gal. = 20 qt. = 40 pt.

⑱ 9 quarters = 45 nickels = 225¢; 6 quarters = 30 nickels = 150¢; 3 quarters = 15 nickels = 75¢

Frank wanted to know about the charts Ruth and Ben might have made when their Mother gave them problems involving subtraction.

"I will show you examples. But I want to warn you — we were not always able to fill in all the boxes in some of our charts. We drew large 'X's' in such boxes.

"Mother might say

3 dimes − 6 cents = 24 cents.

"Then Ben and I would make a chart of many ways of completing the sentence:

____ dimes − ____ cents = ... ¢

"We would outline the answer to Mother's problem with a heavy line."

Plan E.

—	0¢	3¢	6¢	9¢	12¢
1 dime	10¢	7¢	4¢	1¢	X
2 dimes	20¢	17¢	14¢	11¢	8¢
3 d	30¢	27¢	24¢	21¢	18¢
4 d	40¢	37¢	34¢	31¢	28¢
5 d	50¢	47¢	44¢	41¢	38¢

Jumping around.

—	7¢	14¢	25¢	31¢	18¢	42¢
2d	13¢	6¢	X	X	2¢	X
3d	23¢	16¢	5¢	X	12¢	X
5d	43¢	36¢	25¢	19¢	32¢	8¢
8d	73¢	66¢	55¢	49¢	62¢	38¢
6d	53¢	46¢	35¢	29¢	42¢	18¢
9d	83¢	76¢	65¢	59¢	72¢	48¢
10d	93¢	86¢	75¢	69¢	82¢	58¢
11d	103¢	96¢	85¢	79¢	92¢	68¢

"Mother might say

2 hours − 15 minutes = 105 min

and we included the answer as one of the entries in our chart. We can use 'm' as a shorthand for 'minutes.'"

Plan F.

—	15m	30m	45m	60m
1 h	45m	30m	15m	0m
2 h	105m	90m	75m	60m
3 h	165m	150m	135m	120m
4 h	225m	210m	195m	180m

Jumping around.

—	20m	35m	53m	28m
1 h	40m	25m	7m	32m
1½ h	70m	55m	37m	62m
½ h	10m	X	X	2m
2½ h	130m	115m	97m	122m

"Mother might say

3 weeks − 8 days = 13 days.

"We might give her the following answers."

Plan G.

—	2d	5d	8d	11d	14d
0 w	X	X	X	X	X
1 w	5d	2d	X	X	X
2 w	12d	9d	6d	3d	0d
3 w	19d	16d	13d	10d	7d

Jumping around.

—	7d	12d	18d	9d	23d
2 w	7d	2d	X	5d	X
4 w	21d	16d	10d	19d	5d
6 w	35d	30d	24d	33d	19d
10 w	63d	58d	52d	61d	47d

Tasks: *Pupils use the clues provided to help decide which computations to perform and how results are to be recorded in the blue regions.*

Purpose: To perfect computational skills with various units of measure by exploring the *Arithmetic of Things and Numbers.*

Unifying Ideas: Numeration; addition and subtraction; measurement.

The Lesson: Charts for subtraction situations will have some blank boxes unless negative numbers are used.

The charts Ben and his sister begin in each of the examples on this page could grow indefinitely. In Notes on Pupil Page 18 we suggested ways to limit problems so they can be completed. Another way to limit problems is to give a little more information. Instead of exploring ways to make a true statement from:

____ dimes − _____ cents = ... cents

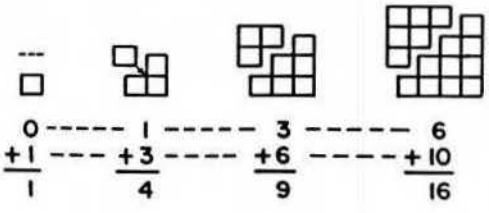
the number of possibilities can be limited by giving the number of dimes:

3 dimes − ____ cents = _____ cents.

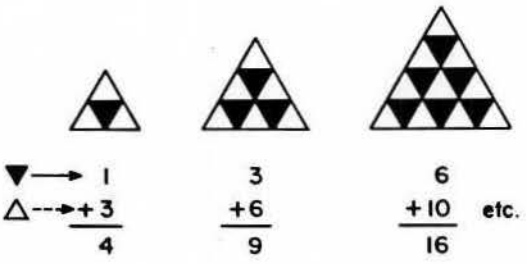
Now all pairs of numbers that give true statements can be listed:

———	0	1	2	...	27	28	29	30
———	30	29	28		3	2	1	0

A quick look at the sketches at the top of practice sheet 19 might lead one to expect to encounter the triangular numbers: 0, 1, 3, 6, 10, 15, etc., again. It may be a surprise that this sequence of sketches leads to the square numbers. In Notes on Pupil Page 17 we noted that finding the sums of successive pairs of triangular numbers gives us the square numbers. We can demonstrate this visually with the stairstep representations of triangular numbers:



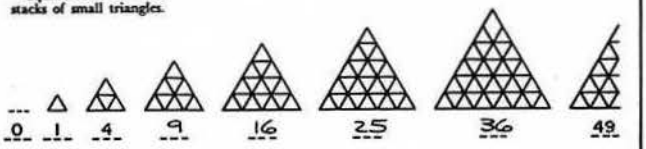
or by coloring the sketches on practice sheet 19 in a certain way:



On practice sheet 17 pupils were asked to express certain numbers as the sums of four square numbers. On practice sheet 19 they are asked to express certain numbers as the sum of four square numbers in three to six different ways. The order in which pupils record these alternatives is of course, not significant. You may wish to ask some pupils to find all possible ways to express 96 as the sum of four square numbers. (96 = 64 + 16 + 16 + 0 is the only way.)

PS-19

Square Numbers — as stacks of small triangles.



0 1 4 9 16 25 36 49

Can you extend the sequence?

0 1 4 9 16 25 36 49 64 81 100 121 144 169 196 225

Use only different combinations of numbers from the above group of numbers to make true statements below.

18 = 16 + 1 + 1 + 0
18 = 9 + 4 + 4 + 1
18 = 9 + 9 + 0 + 0

36 = 36 + 0 + 0 + 0
36 = 9 + 9 + 9 + 9
36 = 25 + 9 + 1 + 1
36 = 16 + 16 + 4 + 0

50 = 49 + 1 + 0 + 0
50 = 36 + 9 + 4 + 1
50 = 25 + 25 + 0 + 0
50 = 25 + 16 + 9 + 0
50 = 16 + 16 + 9 + 9

99 = 81 + 9 + 9 + 0
99 = 81 + 16 + 1 + 1

27 = 25 + 1 + 1 + 0
27 = 9 + 9 + 9 + 0
27 = 16 + 9 + 1 + 1

42 = 36 + 4 + 1 + 1
42 = 16 + 16 + 9 + 1
42 = 25 + 16 + 1 + 0
42 = 25 + 9 + 4 + 4

82 = 36 + 36 + 9 + 1
82 = 64 + 9 + 9 + 0
82 = 81 + 1 + 0 + 0
82 = 49 + 16 + 16 + 1
82 = 49 + 25 + 4 + 4

99 = 64 + 25 + 9 + 1
99 = 49 + 49 + 1 + 0

33 = 16 + 16 + 1 + 0
33 = 16 + 9 + 4 + 4
33 = 25 + 4 + 4 + 0

45 = 25 + 16 + 4 + 0
45 = 36 + 4 + 4 + 1
45 = 36 + 9 + 0 + 0
45 = 16 + 16 + 9 + 4

90 = 81 + 9 + 0 + 0
90 = 64 + 25 + 1 + 0
90 = 64 + 16 + 9 + 1
90 = 81 + 4 + 4 + 1
90 = 49 + 36 + 4 + 1

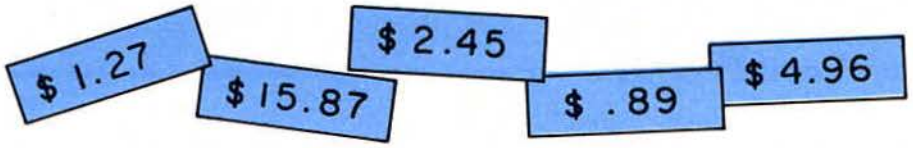
99 = 49 + 25 + 25 + 0
99 = 49 + 25 + 16 + 9

Debby earns 65¢ a week. She saves all but 20¢ of it. She made a chart of her savings:

1 week 45¢	6 weeks $2.70
2 weeks 90¢	7 weeks $3.15
3 weeks $1.35	8 weeks $3.60
4 weeks $1.80	9 weeks $4.05
5 weeks $2.25	10 weeks $4.50

After 6 weeks of saving, she needs to save $2.30 more before she would have $5.00. She would have to save for about 5 more weeks before she would have that much money.

Use your chart results to complete the next sentence: She would have $5.85 in 13 weeks; $7.65 in 17 weeks.



Ben told the class about a puzzle his mother had made up.

"She would give us the price of several items — such as the five given above. If you bought 2 different items, how many different amounts could you spend?

"In this case, I think there are ten different ways. Can you find them all and name the amounts?"

A.

1.	2.	3.	4.	5.
$ 1.27	$ 15.87	$ .89	$ 1.27	$ 1.27
.89	2.45	4.96	15.87	2.45
$ 2.16	$18.32	$ 5.85	$17.14	$ 3.72

6.	7.	8.	9.	10.
$ 1.27	$ 15.87	$ 15.87	$ 2.45	$ 2.45
4.96	.89	4.96	.89	4.96
$ 6.23	$ 16.76	$20.83	$ 3.34	$ 7.41

"Then Mother might ask if we could find as many different amounts buying 3 items at a time."

B.

11.	12.	13.	14.	15.
$ 1.27	$ 1.27	$ 1.27	$ 1.27	$ 1.27
15.87	15.87	15.87	2.45	2.45
2.45	.89	4.96	.89	4.96
$19.59	$ 18.03	$22.10	$ 4.61	$ 8.68

16.	17.	18.	19.	20.
$ 1.27	$ 2.45	$ 15.87	$ 15.87	$ 15.87
.89	.89	.89	2.45	2.45
4.96	4.96	4.96	4.96	.89
$ 7.12	$ 8.30	$ 21.72	$ 23.28	$ 19.21

C.

How does the problem change if you buy 1 item at a time? . . . 4 items at a time? . . . Let's explore these problems.

20-21 Columnar addition with two-, three-, and four-digit numbers

Tasks: In exercise A pupils list the ten possible pairs of items and compute the sum of the prices for each pair. In exercise B pupils list the ten possible triplets of items and compute the sum of the prices for each.

Purpose: To explore strategies for forming combinations of numbers while perfecting computational skills with decimal notation.

Unifying Ideas: Numeration; addition and subtraction; measurement.

The Lesson: In standardized tests and in older textbooks, sale signs or price lists often serve as the basis for problems such as "If Mary bought a dress and two pairs of shoes, how much did she spend?" Such questions lead to an isolated computation that makes no contribution to any larger problem. By contrast, each of the "price-list" computations on pupil page 20 represents a single building block in the solution of a central problem. Pupils are asked to find the different ways two items out of five can be purchased. To do this, a pupil must develop a plan to find all possible sums of two numbers (disregarding the order of the addends). He might label each item with a letter and (if the same letters in a different order are not considered a different pair) make the following list:

AB	AC	AD	AE
	BC	BD	BE
		CD	CE
			DE

Or, he might write all the two-digit numerals he can with the digits 1, 2, 3, 4, 5, with the further restriction that the first digit of each numeral must be smaller than the second (since 12 and 21 represent the purchase of the same pair of items):

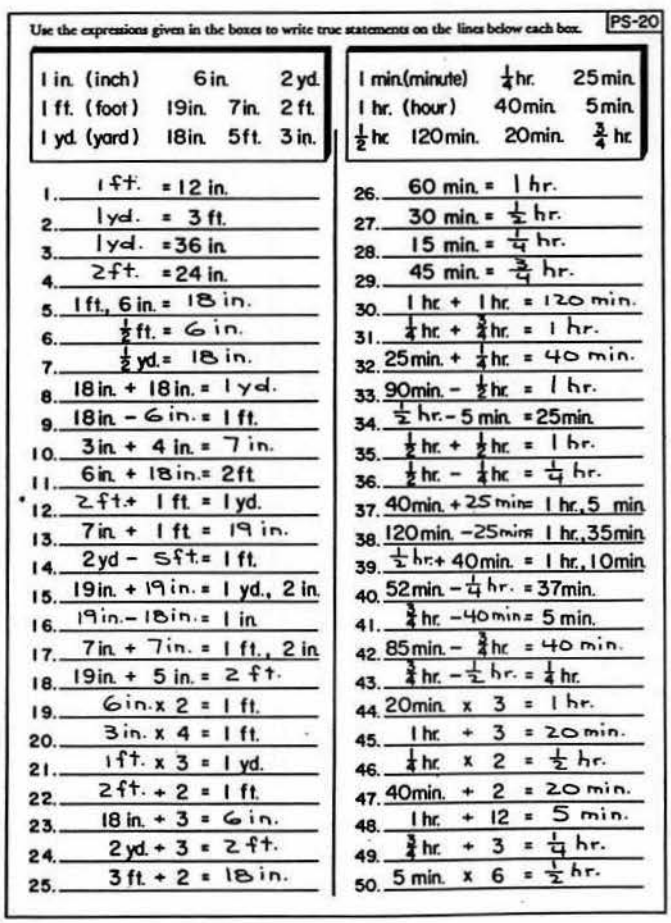
12, 13, 14, 15, 23, 24, 25, 34, 35, and 45

When the pupil has determined the ten purchases, he must then find the amount spent for each. The order of the purchases and of the items in each purchase will, of course, vary widely.

A simple way to list the three-item purchases is to note the three items left out in the list of two-item purchases:

(AB) CDE	(AC) ABD	(AD) BCE	(AE) BCD
	(BC) ADE	(BD) ACE	(BE) ACD
		(CD) ABE	(CE) ABD
			(DE) ABC

Practice sheet 20, the first of a three-page development of examples from the *Arithmetic of Things and Numbers*, is concerned with systems for measuring length (inches, feet, and yards) and time (minutes and hours). Pupils should not need rules for working with the fractions involved; ½ foot can easily be replaced by 6 inches; ¼ hour by 15 minutes; ¾ hours by 45 minutes, etc. Similarly, ½ hr. − ¼ hr. can be replaced by 30 minutes − 15 minutes; ¾ hour ÷ 3 by 45 minutes ÷ 3; ¼ hour ×2 by 15 minutes ×2. Pupils familiar with fractions may be permitted to use them to obtain answers more quickly, but we do not intend to begin any formal discussion of fractions here. We except most pupils to think in terms of whole numbers of smaller units.



Use the expressions given in the boxes to write true statements on the lines below each box. PS-20

1 in. (inch)	6 in.		2 yd.
1 ft. (foot)	19 in.	7 in.	2 ft.
1 yd. (yard)	18 in.	5 ft.	3 in.

1 min. (minute)	¼ hr.		25 min.
1 hr. (hour)	40 min.		5 min.
½ hr.	120 min.	20 min.	¾ hr.

1. 1 ft. = 12 in.
2. 1 yd. = 3 ft.
3. 1 yd. = 36 in.
4. 2 ft. = 24 in.
5. 1 ft., 6 in. = 18 in.
6. ½ ft. = 6 in.
7. ½ yd. = 18 in.
8. 18 in. + 18 in. = 1 yd.
9. 18 in. − 6 in. = 1 ft.
10. 3 in. + 4 in. = 7 in.
11. 6 in. + 18 in. = 2 ft
12. 2 ft. + 1 ft. = 1 yd.
13. 7 in. + 1 ft = 19 in.
14. 2 yd − 5 ft. = 1 ft.
15. 19 in. + 19 in. = 1 yd., 2 in.
16. 19 in. − 18 in. = 1 in.
17. 7 in. + 7 in. = 1 ft., 2 in.
18. 19 in. + 5 in. = 2 ft.
19. 6 in. × 2 = 1 ft.
20. 3 in. × 4 = 1 ft.
21. 1 ft. × 3 = 1 yd.
22. 2 ft. × 2 = 1 ft.
23. 18 in. ÷ 3 = 6 in.
24. 2 yd. ÷ 3 = 2 ft.
25. 3 ft. ÷ 2 = 18 in.
26. 60 min. = 1 hr.
27. 30 min. = ½ hr.
28. 15 min. = ¼ hr.
29. 45 min. = ¾ hr.
30. 1 hr. + 1 hr. = 120 min.
31. ¼ hr. + ¾ hr. = 1 hr.
32. 25 min. + ¼ hr. = 40 min.
33. 90 min. − ½ hr. = 1 hr.
34. ½ hr. − 5 min. = 25 min.
35. ½ hr. + ½ hr. = 1 hr.
36. ½ hr. − ¼ hr. = ¼ hr.
37. 40 min. + 25 min. = 1 hr., 5 min.
38. 120 min. − 25 min. = 1 hr., 35 min.
39. ½ hr. + 40 min. = 1 hr., 10 min.
40. 52 min. − ¼ hr. = 37 min.
41. ¾ hr. − 40 min. = 5 min.
42. 85 min. − ¾ hr. = 40 min.
43. ¾ hr. − ½ hr. = ¼ hr.
44. 20 min. × 3 = 1 hr.
45. 1 hr. ÷ 3 = 20 min.
46. ¼ hr. × 2 = ½ hr.
47. 40 min. ÷ 2 = 20 min.
48. 1 hr. ÷ 12 = 5 min.
49. ¾ hr. ÷ 3 = ¼ hr.
50. 5 min. × 6 = ½ hr.

C.

one at a time . . .

1.	2.	3.	4.	5.
$ 1.27	$15.87	$2.45	$ .89	$4.96

D.

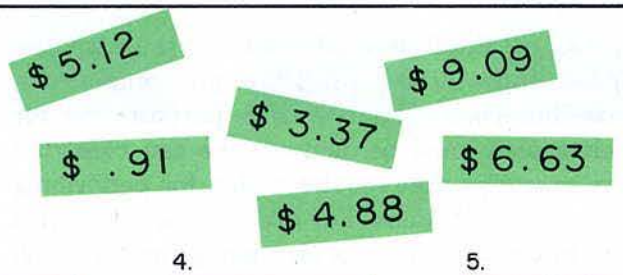
four at a time . . .

1.	2.	3.	4.	5.
$ 1.27	$ 1.27	$ 1.27	$ 1.27	$15.87
15.87	15.87	15.87	2.45	2.45
2.45	2.45	.89	.89	.89
.89	4.96	4.96	4.96	4.96
$ 20.48	$24.55	$22.99	$ 9.57	$24.17

E. "Here is another example.

"Can you find 15 combinations of the prices of the 6 items shown — taking them 4 different prices at a time?

"How many totals are different?"



1.	2.	3.	4.	5.
$ 5.12	$ 5.12	$ 5.12	$ .91	$ 4.88
.91	4.88	4.88	9.09	.91
3.37	.91	3.37	3.37	3.37
4.88	9.09	6.63	6.63	6.63
$ 14.28	$ 20.00	$ 20.00	$ 20.00	$ 15.79

6.	7.	8.	9.	10.
$ 5.12	$ 5.12	$ 5.12	$ 4.88	$ 4.88
.91	9.09	.91	9.09	9.09
6.63	6.63	9.09	3.37	.91
4.88	4.88	3.37	6.63	6.63
$ 17.54	$ 25.72	$ 18.49	$ 23.97	$ 21.51

11.	12.	13.	14.	15.
$ 4.88	$ 5.12	$ 5.12	$ 3.37	$ 5.12
9.09	9.09	9.09	5.12	4.88
.91	3.37	.91	.91	9.09
3.37	6.63	6.63	6.63	3.37
$ 18.25	$ 24.21	$ 21.75	$ 16.03	$ 22.46

Notes on Pupil Page 21

Tasks: In exercises C and D pupils continue the activities initiated on pupil page 20. In exercise E pupils find fifteen different ways that four items out of six can be purchased and compute the amount spent for each purchase.

Purpose: To explore strategies for forming combinations of numbers while perfecting computational skills with decimal notation.

Unifying Ideas: Structure; numeration; addition and subtraction; measurement.

The Lesson: Pupil page 21 extends the consideration of five items at different prices. "One at a time" is obvious—simply list the items one at a time. "Four at a time" is the same as "leaving out one at a time." A shortcut to finding the sums for four at a time is to find the sum for all five and subtract the price of each item from this total.

1.27	$25.44	$25.44	$25.44
15.87	−4.96	−.89	−1.27
2.45	$20.48	$24.55	$24.17
.89			
4.96	$25.44	$25.44	
25.44	−2.45	−15.87	
	$22.99	$ 9.57	

In exercise E four items must be selected from a list of six in fifteen different ways. This may be considered the same problem as excluding two items from a list of six in fifteen different ways.

The prices of the six items were selected so that clever pupils could devise a shortcut:

$ 5.12	$ 9.09	$ 3.37
4.88	.91	6.63
$10.00	$10.00	$10.00

Each combination of four items must contain at least one of these pairs. Three of the fifteen selections contain two of these pairs, so these sums are $20.00. All the other sums are different, making thirteen different sums in all.

Problems like those on pupil pages 20 and 21 can be constructed very easily—with items chosen at random or with built-in relationships. In any case the required computations lead to a conclusion—they are not ends in themselves.

Practice sheet 21 extends the investigation begun on practice sheet 20 to consideration of money and our system for measuring liquids.

Activities similar to those on practice sheets 20 and 21 can easily be built from other lists like those at the top of the practice sheets. One way to develop such a list is to ask pupils to contribute several true statements about measurements.

1 dime + 8 nickels = 1 half dollar
1 quarter + 7 dimes = 95¢
etc.

As their statements are recorded, any new expressions in the statements are included in the list. Finally, the statements are erased—leaving the list. Now, have the pupils construct as many true statements as they can, using only the expressions from the list.

Use the expressions given in the boxes to write true statements on the lines below each box. PS-21

$1.00	73¢	25¢	75¢
$7.83	49¢	$4.50	27¢
$1.75		$1.50	84¢
37¢		50¢	99¢

1 qt (quart)	2qt.	3 pt
1 pt. (pint)	16pt.	3 qt
1 gal. (gallon)		7 gal.
8 gal.	3 gal.	6 qt

1. 9¢ + 18¢ = 27¢
2. $3.00 − $2.50 = 50¢
3. 73¢ + 27¢ = $1.00
4. 75¢ + 75¢ = $1.50
5. $2.00 − $1.25 = 75¢
6. 86¢ − 27¢ = 59¢
7. $1.50 + $4.50 = $6.00
8. 37¢ + 49¢ = 86¢
9. $1.18 + 75¢ = $1.93
10. 49¢ + 75¢ = $1.24
11. $1.00 − 73¢ = 27¢
12. 50¢ x 2 = $1.00
13. 25¢ x 3 = 75¢
14. $2.00 ÷ 4 = 50¢
15. 25¢ x 7 = $1.75
16. 75¢ x 2 = $1.50
17. $1.50 ÷ 2 = 75¢
18. 50¢ x 9 = $4.50
19. 84¢ ÷ 2 = 42¢
20. $4.50 ÷ 3 = $1.50
21. 75¢ ÷ 3 = 25¢
22. 2pt. = 1qt.
23. 4qt. = 1gal.
24. 4pt. = 2qt.
25. 1qt., 1pt. = 3pt.
26. ½gal. = 2qt.
27. 2gal. = 16pt.
28. 1gal. + 2qt. = 6qt.
29. 7gal. + 8gal. = 15gal.
30. 2gal. − 2qt. = 6qt.
31. 1gal. − 3qt. = 1qt.
32. 3qt. − 1pt. = 5pt.
33. 1qt x 4 = 1gal.
34. 1gal. ÷ 2 = 2qt.
35. 9qt ÷ 3 = 3qt.
36. ½pt x 2 = 1pt.
37. 3gal ÷ 2 = 6qt.
38. ½pt x 6 = 3pt.
39. 3pt. x 4 = 6qt.
40. ½gal x 3 = 6qt.
41. 2qt ÷ 4 = 1pt.
42. 16pt. ÷ 4 = 1gal.

"Here is another kind of puzzle Mother invented for us," Ben exclaimed. "It's a kind of chain reaction. We might begin with 4 periods of time. We are to find the difference between each pair of given time periods.

"We start with:

316 minutes | 243 min. | 207 min. | 86 min.

"We remember we are talking about minutes, and we find the difference between each pair. Can you find all 6 pairs?

A.

1.	2.	3.	4.	5.	6.
316	243	316	243	316	207
243	86	207	207	86	86
73	157	109	36	230	121

"Now, we have 6 new periods of time — given in numbers of minutes. Next, we consider all possible pairs of these new time periods — and find the difference in each case. Can you find 15 pairs?

B.

1.	2.	3.	4.	5.
157	230	157	157	230
73	36	109	36	157
84	194	48	121	73

6.	7.	8.	9.	10.
157	230	230	230	121
121	73	109	121	73
36	157	121	109	48

11.	12.	13.	14.	15.
121	121	109	109	73
109	36	73	36	36
12	85	36	73	37

"With a twinkle in her eyes, Mother would ask if we wanted to keep going . . . that is, she asked if we wanted to consider each pair of the 11 new periods of time.

"We were anxious to go on. Perhaps you are, also. If you do go on, let me warn you we found more than 50 pairs to consider."

(59, TO BE EXACT.)

22-23 Subtraction with two-, three-, and four-digit numbers

Tasks: In exercise A pupils list six pairs of time periods and find the difference between the members of each pair. In exercise B pupils list fifteen pairs of the six differences found in exercise A and find the difference between the members of each of the fifteen pairs.

Purpose: To explore strategies for forming combinations of numbers while perfecting computational skills with whole numbers.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: Pupil page 22 continues the activities of pupil pages 20 and 21. These are computation pages—but computation with a purpose and for which a plan must be devised. We can construct many examples that require a plan for performing a list of computations. For example, we can get a lot of information from a sign along the highway:

EAST on Route 4 →	
	Miles
Afton	9
Brookdale	26
Centerville	53
Dawson	110
Elburn	134

One way to summarize this information is to construct a mileage chart:

	Afton	Brookdale	Centerville	Dawson
Brookdale	17			
Centerville	44	27		
Dawson	101	84	57	
Elburn	125	108	81	24

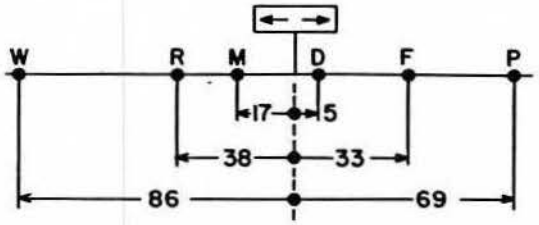
Distance in Miles

Here is another example, a pair of signs at one point on a highway:

Malta	17←
Rand	38←
Westfield	86←

Dallon	5→
Frankford	33→
Paxton	69→

A sketch might clarify the situation:



From this we can make a mileage chart:

	W	R	M	D	F
R	48				
M	69	21			
D	91	43	22		
F	119	71	50	28	
P	155	107	86	64	36

Practice sheet 22 provides further work with the four systems of measurement considered on practice sheets 20 and 21.

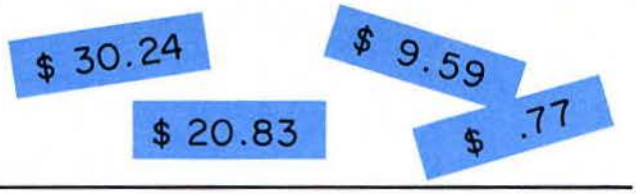
PS-22

Use the expressions given in the box to write true statements on the lines below the box.

1 yd.(yard)	24 ¢	$1.00			
1 gal.(gallon)	2 ft. (feet)				
40 ¢	7 ft.	1 qt.(quart)			
1 pt. (pint)	2 yd.	19 ¢	1 ft.	16 ¢	19 ¢	3 in.
9 in.	1 ft, 6 in.	2 qt.	3 gal.	1 hr. (hour)		25 ¢
50 ¢	5 gal.	10 min.(minutes)	27 ¢	5 pt.		15 min.
2 pt.	44 ¢	45 min.	$1.06	9 hr.	26 ¢	2 gal.
½ hr.	53 ¢	9 ¢	5 hr.	12 min.	50 ¢	20 min.
3 pt.	6 hr.	36 min.	6 in.	4 pt.	$5.00	36 hr.	15 hr.

1.	5 hr. + 1 hr. = 6 hr.	19.	9 in. × 4 = 1 yd.
2.	1 yd. − 1 ft. = 2 ft.	20.	2 qt. × 2 = 1 gal.
3.	24¢ + 16¢ = 40¢	21.	9 in. × 2 = 1 ft, 6 in.
4.	3 in + 9 in. = 1 ft.	22.	$1.00 ÷ 2 = 50¢
5.	1 qt. + 3 qt. = 1 gal.	23.	3 in. × 4 = 1 ft.
6.	7 ft. − 1 ft. = 2 yd.	24.	1 hr. ÷ 2 = ½ hr.
7.	19¢ + 25¢ = 44¢	25.	50¢ × 2 = $1.00
8.	1 qt. + 2 pt. = 2 qt.	26.	45 min. ÷ 3 = 15 min.
9.	5 gal. − 3 gal. = 2 gal.	27.	1 gal. ÷ 4 = 1 qt.
10.	10 min. + 20 min. = ½ hr.	28.	12 min. × 3 = 36 min.
11.	53¢ − 27¢ = 26¢	29.	27¢ ÷ 3 = 9¢
12.	½ hr. + 15 min. = 45 min.	30.	6 in. × 3 = 1 ft, 6 in.
13.	53¢ + 53¢ = $1.06	31.	5 hr. × 3 = 15 hr.
14.	3 pt. + 1 pt. = 2 qt.	32.	50¢ × 10 = $5.00
15.	1 hr − 15 min. = 45 min.	33.	36 hr. ÷ 4 = 9 hr.
16.	9 hr. + 6 hr. = 15 hr.	34.	15 hr. ÷ 3 = 5 hr.
17.	1 ft, 6 in. + 6 in. = 2 ft.	35.	50¢ ÷ 2 = 25¢
18.	3 pt. − 1 qt. = 1 pt.	36.	12 min. × 5 = 1 hr.

"Here's one I made up," Ruth told the class. "It starts out with 4 different amounts of money to compare."



A. Differences.

1.	2.	3.	4.	5.	6.
$ 30.24	$ 30.24	$30.24	$20.83	$20.83	$ 9.59
9.59	20.83	.77	9.59	.77	.77
$ 20.65	$ 9.41	$29.47	$ 11.24	$20.06	$ 8.82

B. Can you find 15 pairs of the differences to consider?

1.	2.	3.	4.	5.
$ 29.47	$ 29.47	$ 29.47	$ 29.47	$ 29.47
20.65	20.06	11.24	9.41	8.82
$ 8.82	$ 9.41	$ 18.23	$ 20.06	$ 20.65

6.	7.	8.	9.	10.
$ 20.65	$ 20.65	$ 20.65	$ 20.65	$ 9.41
20.06	11.24	9.41	8.82	8.82
$ .59	$ 9.41	$ 11.24	$ 11.83	$ .59

11.	12.	13.	14.	15.
$ 20.06	$ 20.06	$ 20.06	$ 11.24	$ 11.24
11.24	9.41	8.82	9.41	8.82
$ 8.82	$ 10.65	$ 11.24	$ 1.83	$ 2.42

"I notice," Ruth pointed out, "that some of the differences are repeated . . . in fact, there are 4 that are repeated. Those repeated are:

C.

1.	2.
$ 8.82	$ 9.41
3.	4.
$ 11.24	$ .59

"And, I notice that 3 of these were in the first list of 6 differences. Do you suppose this is an accident?

"In addition to the 4 repeats, there are 7 other differences.

D.

1.	2.	3.
$ 18.23	$ 20.06	$ 20.65
4.	5.	6.
$ 11.83	$ 10.65	$ 1.83
	7.	
	$ 2.42	

"You might like to consider all pairs of these differences — the 4 repeats and the 7 others . . . 11 in all."

Tasks: In exercise A pupils list the six possible pairs of prices for which differences can be found and find these differences. In exercise B pupils list the fifteen pairs of the differences found in exercise A for which differences can be found and find these differences. In exercise C pupils list the four differences that occurred more than once among the fifteen found in exercise B. In exercise D pupils list those differences that occurred only once among the fifteen found in exercise B.

Purpose: To explore strategies for forming combinations of numbers while perfecting computational skills with decimals.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: Many situations can lead to sets of differences such as those on pupil page 23 and the three preceding pupil pages:

Heights of various children

Amounts saved by several people

Amounts saved by one person week-by-week over a period of several weeks

The number of pages in several books

Dates in history

Times during the day

Problems arising from such lists of information require a great deal of computation, but only as a means to an end. In each, the pupil must think his way through the situation, organizing the information given and the information he finds by computing.

On practice sheet 23 pupils test whole numbers greater than 1 and assign them to one or the other of two mutually exclusive sets. The test is: Take a certain number of tiles and try to arrange them in a rectangular form that has at least two rows and two columns. If this can be done the number goes in one list; if not it goes in the other. Of course, 4 is the smallest number of tiles that can be arranged in such a rectangular form, so 2 and 3 go in one list and 4 in the other.

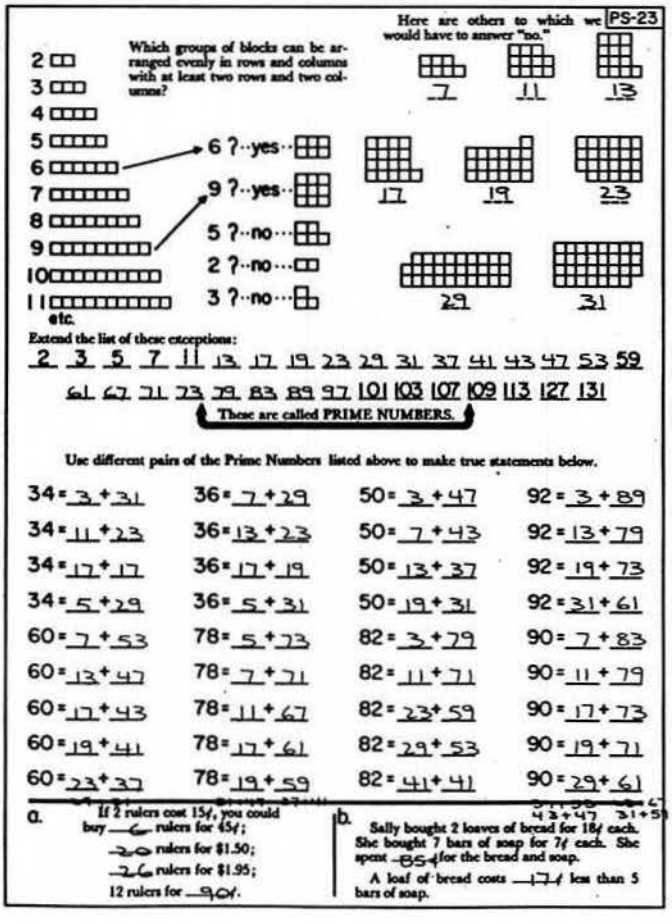
This leads to labeling each whole number greater than 1 as a prime number or a composite number. The composite numbers are those that yield the kind of rectangular arrangement described above (that is, numbers that can be expressed as the product of two factors, each greater than 1). The remaining numbers are the prime numbers (those that can be expressed only as the product of themselves and 1). The number 1 is excluded from both lists, primarily so that we can make the statement: Every whole number greater than 1 is a prime number or can be expressed as the product of prime numbers in only one way:

$12 = 2 \times 2 \times 3$ $\quad$ $35 = 5 \times 7$

$22 = 2 \times 11$ $\quad$ $40 = 2 \times 2 \times 2 \times 5$

etc.

On practice sheet 23 pupils express eight even numbers as the sum of two prime numbers (the same or different) in four or five different ways. Some students may wish to explore other even numbers. An individual or a group might like to extend the list of prime numbers beyond 131. (There is no largest prime number, so such a list is endless.)


PS-23
Which groups of blocks can be arranged evenly in rows and columns with at least two rows and two columns?
2
3
4
5
6
7
8
9
10
11
etc.
6 ?..yes..
9 ?..yes..
5 ?..no...
2 ?..no...
3 ?..no...
Here are others to which we would have to answer "no."
7
11
13
17
19
23
29
31
Extend the list of these exceptions:
2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59
61 67 71 73 79 83 89 97 101 103 107 109 113 127 131
These are called PRIME NUMBERS.
Use different pairs of the Prime Numbers listed above to make true statements below.
34 = 3 + 31
34 = 11 + 23
34 = 17 + 17
34 = 5 + 29
60 = 7 + 53
60 = 13 + 47
60 = 17 + 43
60 = 19 + 41
60 = 23 + 37
36 = 7 + 29
36 = 13 + 23
36 = 17 + 19
36 = 5 + 31
78 = 5 + 73
78 = 7 + 71
78 = 11 + 67
78 = 17 + 61
78 = 19 + 59
50 = 3 + 47
50 = 7 + 43
50 = 13 + 37
50 = 19 + 31
82 = 3 + 79
82 = 11 + 71
82 = 23 + 59
82 = 29 + 53
82 = 41 + 41
92 = 3 + 89
92 = 13 + 79
92 = 19 + 73
92 = 31 + 61
90 = 7 + 83
90 = 11 + 79
90 = 17 + 73
90 = 19 + 71
90 = 29 + 61
a.
If 2 rulers cost 15¢, you could buy 6 rulers for 45¢;
20 rulers for $1.50;
26 rulers for $1.95;
12 rulers for 90¢.
b.
Sally bought 2 loaves of bread for 18¢ each. She bought 7 bars of soap for 7¢ each. She spent 85¢ for the bread and soap.
A loaf of bread costs 17¢ less than 5 bars of soap.

IV Caramel Company Arithmetic: Base-7 and Base-10 Systems

Arithmetic in a Caramel Company.

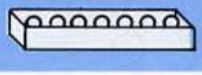
A group of boys organized a Caramel Company. They decided to package the caramels they made in this way:

7 caramels to a package.
7 packages to a box.
7 boxes to a carton.

There are 7 caramels in a package — but how many in a box, and how many in a carton?

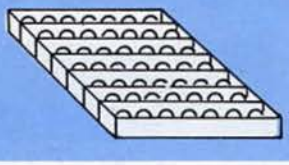
The sketches below may help.

one package:



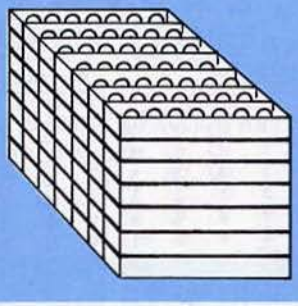
7 caramels

one box:



7 packages
49 caramels

one carton:



7 boxes
49 packages
343 caramels

Tom was in charge of the packaging and shipping department.

To help keep records more easily, he invented a shorthand:

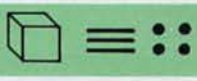
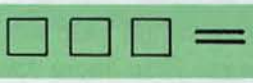
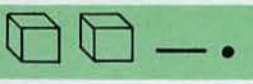
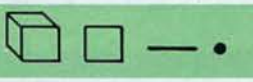
- • ..... a caramel
- — ..... a package of 7 caramels
- □ ..... a box of 7 packages
- ⧉ ..... a carton of 7 boxes

In this shorthand, he could write 1 carton, 2 boxes, 4 packages, and 3 caramels in this way:

Can you find the total number of caramels:

1 carton ..... 343 caramels
2 boxes ..... 98 caramels
4 packages ..... 28 caramels
3 caramels ..... 3 caramels
Total: 472 caramels

Here are other examples of Tom's shorthand. Please fill in the totals:

		Totals
a.		41
b.		368
c.		161
d.		82
e.		694
f.		400
g.		800

Try your hand at Tom's way of keeping records. Use the fewest containers possible.

h.		8
i.		100
j.		59

24-25 Keeping track in multiples of 1, 7, 49, and 343 (a kind of base-7 system of numeration)

Tasks: *Pupils read the story about the Caramel Company's methods of packaging and recording various quantities of caramels. In exercises* a-g *pupils translate from Caramel Company notation to ordinary notation. In exercises* h-j *pupils translate from ordinary notation to Caramel Company notation.*

Purpose: To explore the base-seven system of numeration while perfecting whole-number computational skills in ordinary base-ten notation.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

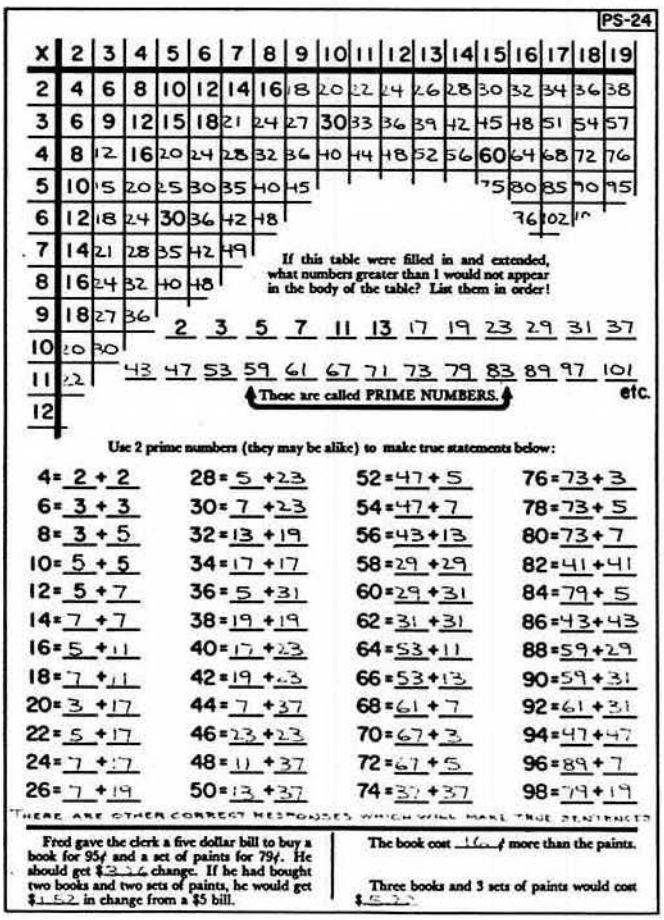
The Lesson: Most of the common units of measure have been encountered in the last eleven pupil pages in the *Arithmetic of Things and Numbers*. In pupil pages 24 through 29 pupils will explore a system used by the Caramel Company to record shipments. The system is based on an unusual method of packaging caramels. This Caramel Company begins by placing 7 caramels in 1 box. This is comparable to 7 days grouped into 1 week. However, our system for measuring time does not proceed systematically from this point. A more systematic plan would be one in which 7 days were grouped into a week, 7 weeks into a month, and 7 months into a year:

1 day = 1 unit
1 day × 7 = 1 week (7 days)
1 week × 7 = 1 month (49 days)
1 month × 7 = 1 year (343 days)

The packaging system chosen by the Caramel Company leads to just such a regular system—a a base-seven system. As we work with this system we will find that records we keep look strange. The thinking we do doesn't change—just the way we record results. Operations (addition, multiplication, subtraction, and division) have the same meaning as before and are performed in much the same way, but answers are recorded differently. On pupil page 30 the packaging system is changed, and the arithmetic looks more familiar.

Practice sheet 24 begins with a portion of a multiplication chart that, extended forever, would include products of all pairs of whole numbers larger than 1. The products in the extended chart would comprise the set of composite numbers. All whole numbers larger than 1 that do not appear as products in the chart are prime numbers.

Many years ago, the mathematician Goldbach said that he thought all even numbers larger than 2 could be expressed as the sum of two primes. All such numbers less than 100 are explored on practice sheet 24. No one has been able to prove that Goldbach was right. Nor has anyone proved him wrong! It is interesting to note that such a simply stated problem has challenged many excellent mathematicians for centuries and is still unsolved.



PS-24

X	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76
5	10	15	20	25	30	35	40	45						75	80	85	90	95
6	12	18	24	30	36	42	48								96	102		
7	14	21	28	35	42	49												
8	16	24	32	40	48													
9	18	27	36															
10	20	30																
11	22																	
12																		

If this table were filled in and extended, what numbers greater than 1 would not appear in the body of the table? List them in order!

2 3 5 7 11 13 17 19 23 29 31 37 43 47 53 59 61 67 71 73 79 83 89 97 101 etc.

These are called PRIME NUMBERS.

Use 2 prime numbers (they may be alike) to make true statements below:

4=2+2
6=3+3
8=3+5
10=5+5
12=5+7
14=7+7
16=5+11
18=7+11
20=3+17
22=5+17
24=7+17
26=7+19
28=5+23
30=7+23
32=13+19
34=17+17
36=5+31
38=19+19
40=17+23
42=19+23
44=7+37
46=23+23
48=11+37
50=13+37
52=47+5
54=47+7
56=43+13
58=29+29
60=29+31
62=31+31
64=53+11
66=53+13
68=61+7
70=67+3
72=67+5
74=37+37
76=73+3
78=73+5
80=73+7
82=41+41
84=79+5
86=43+43
88=59+29
90=59+31
92=61+31
94=47+47
96=89+7
98=79+19

THERE ARE OTHER CORRECT RESPONSES WHICH WILL MAKE TRUE SENTENCES

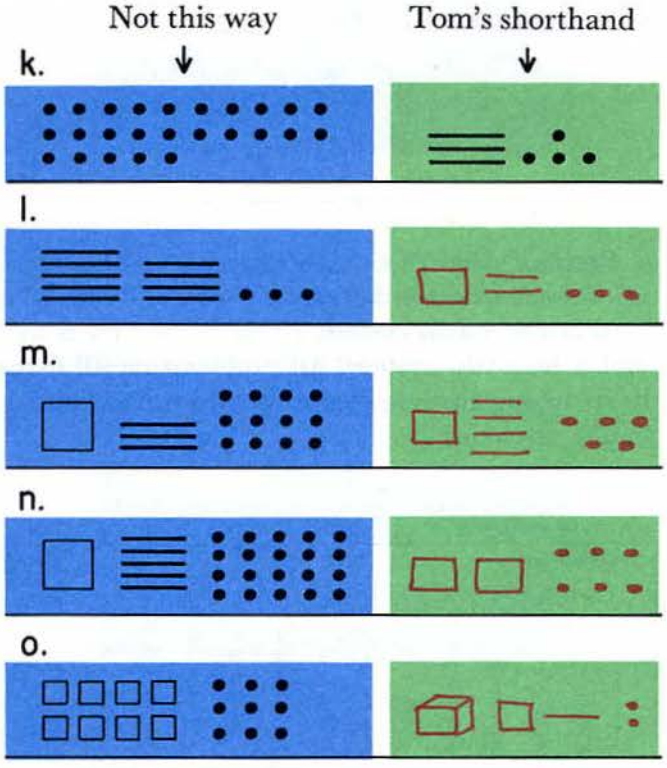
Fred gave the clerk a five dollar bill to buy a book for 95¢ and a set of paints for 79¢. He should get $3.26 change. If he had bought two books and two sets of paints, he would get $1.52 in change from a $5 bill.

The book cost 16¢ more than the paints.

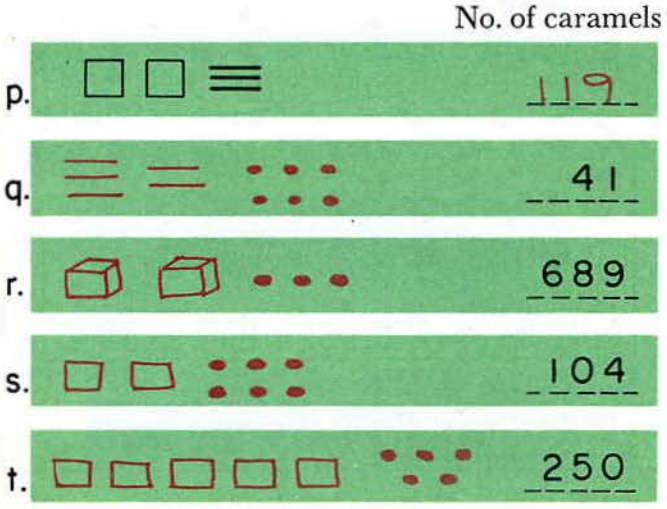
Three books and 3 sets of paints would cost $6.22

Tom explained, "We always fill an order with the fewest cartons, boxes, packages, and separate caramels we can.

"For example: we would not ship 25 separate caramels. We would ship 3 packages and 4 loose caramels. We follow a similar rule in our shorthand."

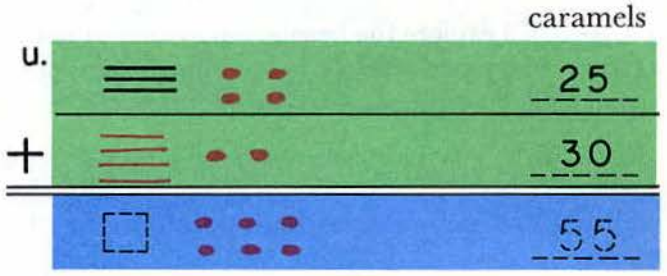


Here is some practice in reading and writing Tom's shorthand:

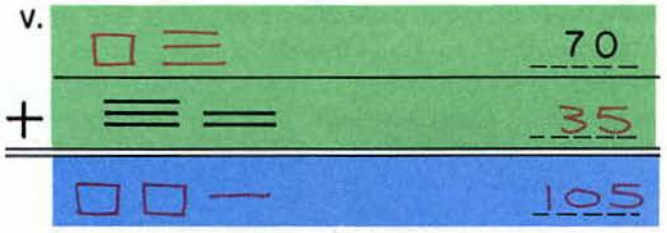


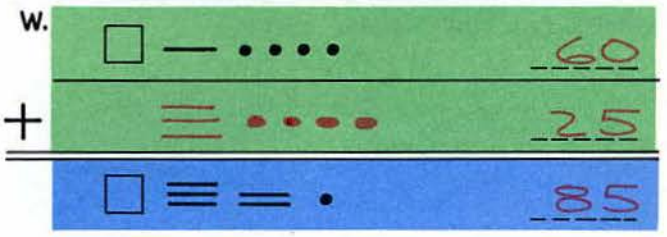
"Sometimes a customer would order some caramels — and then place another order before we had shipped the first order. So we invented a kind of addition.

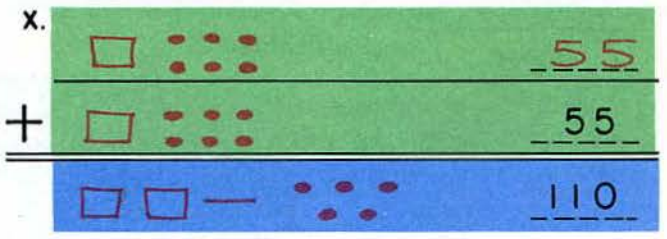
"Suppose the first order was for 25 caramels and the next was for 30 caramels. Our addition looked like this:" (Please complete the examples.)

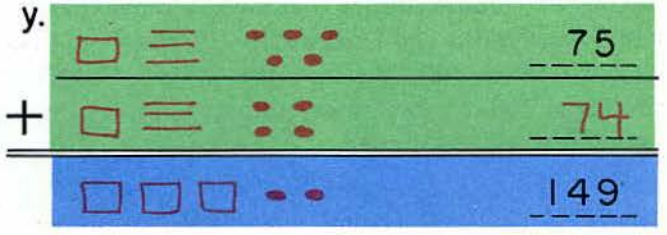


Here are other examples of Tom's "Caramel Company Addition" . . . please complete them.









Tasks: *In exercises* k-o *pupils rewrite the notations on the left in Tom's shorthand notation. In exercise* p *they express in decimal notation the number given in Tom's notation and in exercises* q-t *they translate the decimal notation into Tom's notation. Finally, in exercises* u-y *pupils translate from one notation to the other and perform the indicated computations.*

Purpose: To explore the base-seven system of numeration while performing whole-number computational skills in ordinary base-ten notation.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

The Lesson: The Caramel Company always fills orders with the fewest possible cartons, boxes, packages, and separate caramels. Otherwise, there would be several ways to ship caramels:

Shorthand		Number of Items
○○○○○○○○○○○○○ ○○○○○○○○○○○○	(25)	25
— ○○○○○○○○○ ○○○○○○○○○	(7+18)	19
= ○○○○○○ ○○○○○	(14+11)	13
≡ ○○○○	(21+4)	7

The last line shows how they would ship twenty-five caramels. They will never ship more than 6 separate caramels or 6 separate packages or 6 separate cartons. Thus, in their system, based on grouping by sevens, no shorthand record will indicate more than 6 of any one grouping. Similarly, in our decimal system, based on grouping by tens, we never record more than 9 units or 9 tens or 9 hundreds, etc.

In the *Punch Card Arithmetic* on practice sheet 25, we explore number patterns that have much in common with Caramel Company arithmetic. If the Company put 3 caramels in a package, 3 packages (9 caramels) in a carton, and 3 cartons (27 caramels) in the next larger container, they would ship caramels this way:

(1) ○ (5) $\overline{○\ ○}$ (9) □

(2) ○ ○ (6) = (10) □ ○

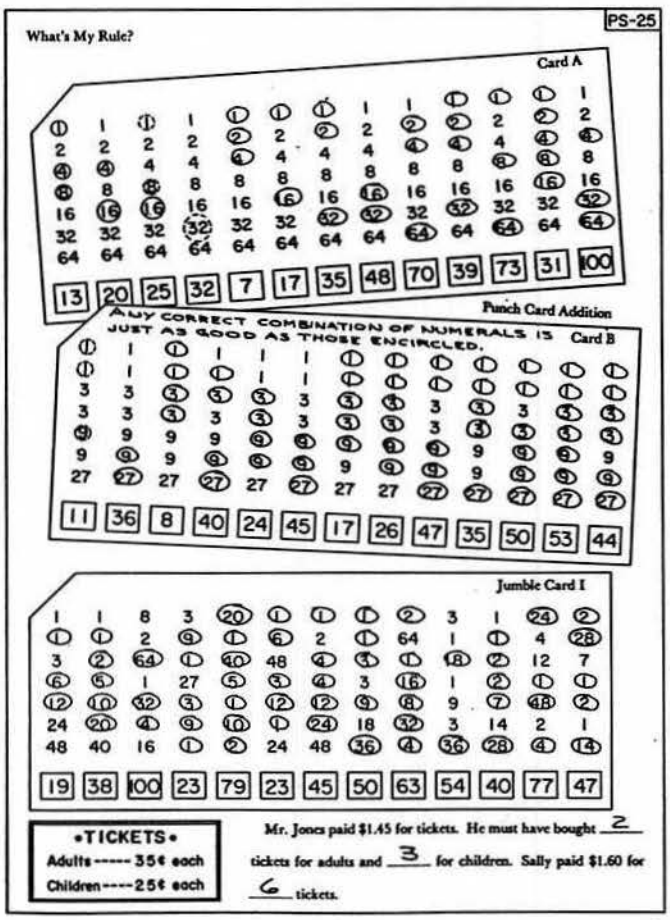
(3) — (7) $\overline{=\ ○}$ (11) □ ○ ○

(4) $\overline{—\ ○}$ (8) $\overline{=\ ○\ ○}$ (12) □ —

etc.

They would ship no more than 2 separate caramels, or 2 packages (of 3 caramels), or 2 cartons (of 3 packages), etc. Thus, the number in any shipment would be recorded as a sum of the numbers 1, 1, 3, 3, 9, 9, 27, 27, etc. All numbers can be indicated by "punching" some unique combination of these numbers:

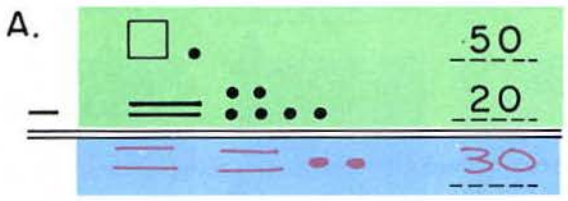
(1)	①	1	3	3	9	9	27	27
(2)	①	①	3	3	9	9	27	27
(3)	1	1	③	3	9	9	27	27
(4)	①	1	③	3	9	9	27	27
(5)	①	①	③	3	9	9	27	27
(6)	1	1	③	③	9	9	27	27
(7)	1	①	③	③	9	9	27	27
(13)	1	①	3	③	⑨	9	27	27
(20)	①	①	3	3	⑨	⑨	27	27
(43)	1	①	③	③	9	⑨	㉗	27


What's My Rule?
PS-25
Card A
13 20 25 32 7 17 35 48 70 39 73 31 100
Punch Card Addition
ANY CORRECT COMBINATION OF NUMERALS IS JUST AS GOOD AS THOSE ENCIRCLED.
Card B
11 36 8 40 24 45 17 26 47 35 50 53 44
Jumble Card I
19 38 100 23 79 23 45 50 63 54 40 77 47
•TICKETS•
Adults ----- 35¢ each
Children ---- 25¢ each
Mr. Jones paid $1.45 for tickets. He must have bought 2 tickets for adults and 3 for children. Sally paid $1.60 for 6 tickets.

"We also invented our special kind of subtraction.

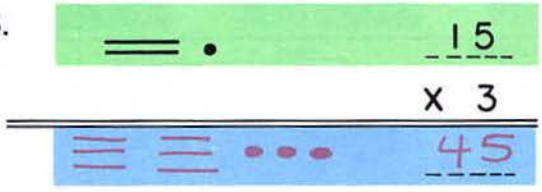
"Suppose a customer ordered 50 caramels . . . and after we had the order packed, the customer said they had ordered 20 too many. Our subtraction would look like this:

A.



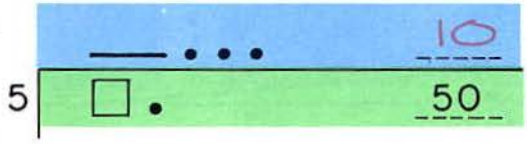
"We also invented a 'Caramel Company Multiplication.' A customer might order 15 caramels . . . then call back asking us to change his order to 3 times as many."

B.



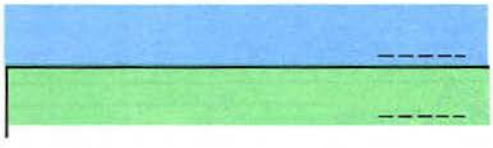
"Sometimes we needed a 'Caramel Company Division.' A customer might order 50 caramels and ask us to send them to his five grandchildren — the same number to each."

C.

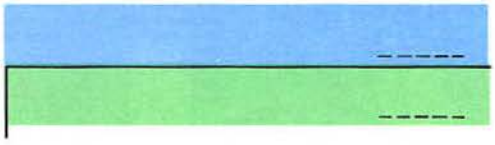


Make up a multiplication and division example of your own.

D.



E.



"This was a helpful shorthand," Tom explained, "but it was clumsy. So we changed to a shorthand we could write more rapidly. Here are some examples of the old and new methods. Perhaps you can fill in the blanks.

	Old way	New way
F.	□□ — ••••	… 0,2,1,,
G.	□□ □□□ ≡ ⁘	… 2,3,5,6
H.	□□□ □ ••••	… 0,5,0,4
I.	□□ •••••	… 2,0,0,5

Here are examples of Tom's new Caramel Company shorthand. Please fill in the missing items.

Addition

J.

1,0,3,1
+ 2,1,1,5
3,1,4,6

K.

0,1,0,1
+ 0,0,2,6
0,1,3,0

Subtraction

L.

0,5,1,4
− 0,1,0,1
0,4,1,3

M.

0,1,0,1
− 0,0,0,4
0,0,6,4

Multiplication

N.

1,3,0,2
X 2
2,6,0,4

O.

0,0,2,6
X 3
0,1,1,4

Division

P.

2,0,3,1
2 | 4,0,6,2

Q.

0,0,4,1
3 | 0,1,5,3

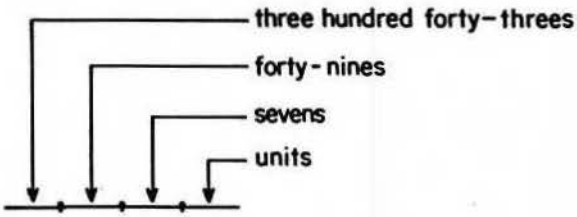
Tasks: In exercises A-C pupils carry out the computations in decimal form and translate the results into Tom's shorthand notation. In exercises D and E pupils construct their own examples. In exercises F-I they translate Tom's shorthand notation into the new notation. Finally, in exercises J-Q they carry out the required computations using the new notation.

Purpose: To explore the base-seven system of numeration while perfecting whole-number computational skills in ordinary base-ten notation.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

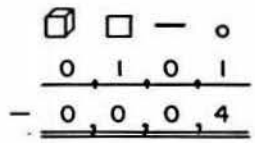
The Lesson: On pupil page 25 we could add two different orders of caramels and find an order of caramels we could call their sum. Similarly, in exercise A of pupil page 26, we subtract one order from another to find the difference. As in the *Arithmetic of Things and Numbers*, we also consider an order of caramels multiplied by a number—doubled, tripled, multiplied by 4, etc.—and divided by a number.

Also, in the interest of simplified record keeping, we introduce an improved shorthand. This is essentially a "place-value" system:

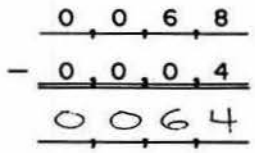


We use zeros to indicate that no container of a given size is used. We could dispense with the commas; however, we will use them throughout to avoid confusion with decimal notation.

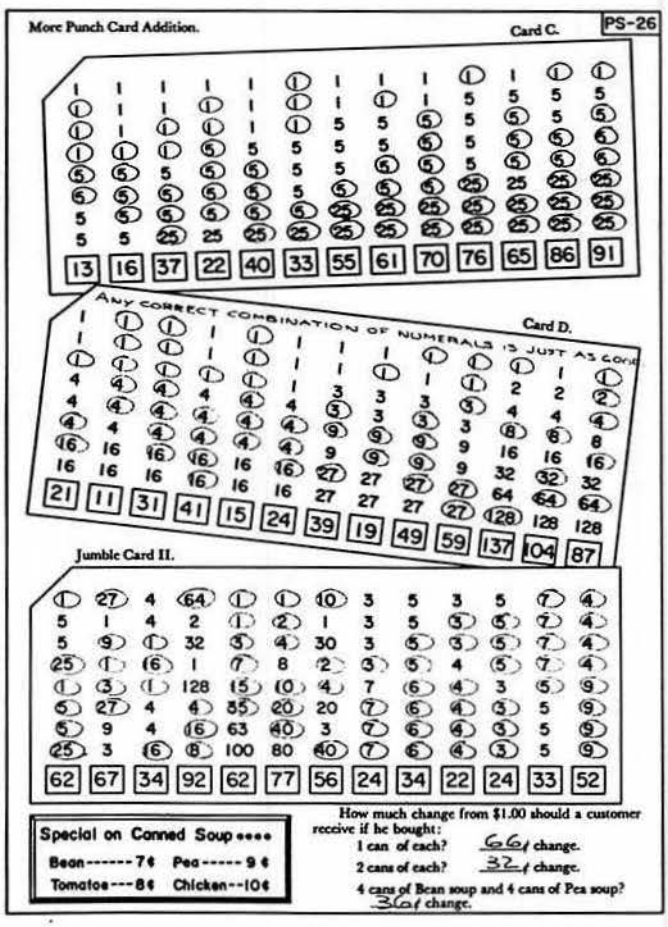
In exercises J-Q in this shorthand, the examples in the left column do not require any repackaging, while those in the right column do. Pupils should be encouraged to talk through the steps of similar problems on the board as preparation for independent work. Example:



Take 4 caramels away from an order of 1 carton and 1 caramel. Solution: Break the carton up into 7 packages. Open one of the packages and add the 7 separate caramels to the one you already have:



Practice sheet 26 extends experience with *Punch Card Arithmetic*. Card C explores a base-five system; card D a base-four system, a base-three system, and and a base-two system. All problems on all cards have unique answers except for which particular 1, 5, etc., is circled.


More Punch Card Addition.
Card C.
PS-26
13 16 37 22 40 33 55 61 70 76 65 86 91
Any correct combination of numerals is just as good.
Card D.
21 11 31 41 15 24 39 19 49 59 137 104 87
Jumble Card II.
62 67 34 92 62 77 56 24 34 22 24 33 52
Special on Canned Soup
Bean------7¢ Pea-----9¢
Tomatoe---8¢ Chicken--10¢
How much change from $1.00 should a customer receive if he bought:
1 can of each? 66¢ change.
2 cans of each? 32¢ change.
4 cans of Bean soup and 4 cans of Pea soup? 36¢ change.

Here are more examples of Caramel Company arithmetic.

Remember there are ..... 7 caramels to a package.
7 packages to a box.
7 boxes to a carton.

a
$$\begin{array}{r} 1,4 \\ +\ 0,5 \\ \hline 2,2 \end{array}$$

b
$$\begin{array}{r} 0,6 \\ +\ 0,6 \\ \hline 1,5 \end{array}$$

c
$$\begin{array}{r} 1,3 \\ +\ 1,4 \\ \hline 3,0 \end{array}$$

g
$$\begin{array}{r} 4,5,1 \\ +\ 1,0,6 \\ \hline 5,6,0 \end{array}$$

h
$$\begin{array}{r} 0,4,3 \\ +\ 0,5,5 \\ \hline 1,3,1 \end{array}$$

d
$$\begin{array}{r} 3,0 \\ -\ 0,5 \\ \hline 2,2 \end{array}$$

e
$$\begin{array}{r} 5,4 \\ -\ 1,3 \\ \hline 4,1 \end{array}$$

f
$$\begin{array}{r} 4,2 \\ -\ 1,4 \\ \hline 2,5 \end{array}$$

i
$$\begin{array}{r} 3,0,5 \\ -\ 0,4,2 \\ \hline 2,3,3 \end{array}$$

j
$$\begin{array}{r} 1,0,0 \\ -\ 0,2,6 \\ \hline 0,4,1 \end{array}$$

1
$$\begin{array}{r} 4,3 \\ +\ 1,4 \\ \hline 6,0 \end{array}$$

2
$$\begin{array}{r} 2,5 \\ +\ 2,1 \\ \hline 4,6 \end{array}$$

3
$$\begin{array}{r} 1,6 \\ +\ 3,6 \\ \hline 5,5 \end{array}$$

4
$$\begin{array}{r} 2,3,4 \\ +\ 0,5,1 \\ \hline 3,1,5 \end{array}$$

5
$$\begin{array}{r} 3,1,4 \\ +\ 1,5,4 \\ \hline 5,0,1 \end{array}$$

6
$$\begin{array}{r} 5,0 \\ -\ 0,2 \\ \hline 4,5 \end{array}$$

7
$$\begin{array}{r} 3,6 \\ -\ 1,2 \\ \hline 2,4 \end{array}$$

8
$$\begin{array}{r} 4,5 \\ -\ 1,6 \\ \hline 2,6 \end{array}$$

9
$$\begin{array}{r} 3,2,5 \\ -\ 0,4,3 \\ \hline 2,5,2 \end{array}$$

10
$$\begin{array}{r} 5,1,2 \\ -\ \ 1,6 \\ \hline 4,6,3 \end{array}$$

Multiplication

11
$$\begin{array}{r} 0,3 \\ \times\ 3 \\ \hline 1,2 \end{array}$$

12
$$\begin{array}{r} 3,2 \\ \times\ 2 \\ \hline 6,4 \end{array}$$

13
$$\begin{array}{r} 1,5 \\ \times\ 2 \\ \hline 3,3 \end{array}$$

14
$$\begin{array}{r} 1,3,2 \\ \times\ 3 \\ \hline 4,2,6 \end{array}$$

15
$$\begin{array}{r} 0,2,1 \\ \times\ 6 \\ \hline 1,5,6 \end{array}$$

16
$$\begin{array}{r} 0,4 \\ \times\ 5 \\ \hline 2,6 \end{array}$$

17
$$\begin{array}{r} 1,2 \\ \times\ 4 \\ \hline 5,1 \end{array}$$

18
$$\begin{array}{r} 2,1 \\ \times\ 3 \\ \hline 6,3 \end{array}$$

19
$$\begin{array}{r} 2,3,1 \\ \times\ 2 \\ \hline 4,6,2 \end{array}$$

20
$$\begin{array}{r} 1,1,5 \\ \times\ 3 \\ \hline 3,5,1 \end{array}$$

21
$$\begin{array}{r} 0,6 \\ \times\ 3 \\ \hline 2,4 \end{array}$$

22
$$\begin{array}{r} 1,4 \\ \times\ 4 \\ \hline 6,2 \end{array}$$

23
$$\begin{array}{r} 0,6 \\ \times\ 6 \\ \hline 5,1 \end{array}$$

24
$$\begin{array}{r} 0,3,2 \\ \times\ 3 \\ \hline 1,2,6 \end{array}$$

25
$$\begin{array}{r} 1,3,4 \\ \times\ 2 \\ \hline 3,0,1 \end{array}$$

Notes on Pupil Page 27

Tasks: *In exercise* a-j *and 1-10 pupils find missing sums and addends. In exercises 11-25 they find missing products and factors.*

Purpose: To explore the base-seven system of numeration while perfecting whole-number computational skills in ordinary base-ten notation.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: For pupils having trouble with the base-seven system, talk through a few examples from the pupil's page. Exercise *a* tells us that 5 caramels are to be added to an order for 1 package and 4 caramels. Without repacking, that would be 1 package and 9 separate caramels. However, 7 of the 9 would be put in a package, making 2 packages and 2 caramels.

In exercise *d* we find the result of removing 5 caramels from a shipment of 3 packages. One package must be opened and 5 of its caramels removed, leaving 2 packages and 2 caramels.

In example 11 an order for 3 caramels is to be tripled. The result would be 9 caramels, 7 of which would be put in a package; the shipment would be 1 package and 2 caramels.

In example 10 we begin with an order of 5 cartons, 1 package, and 2 caramels and, after removing some, ship 4 cartons, 6 packages, and 3 caramels. We must find how many were removed.

10.

$$\begin{array}{r} 5\ 1\ 2 \\ -\ \_\ \_\ \_ \\ \hline 4\ 6\ 3 \end{array}$$

We might begin by noting that we cannot get 3 caramels by removing single caramels from the 2 we have. We must break open a package to have:

$$\begin{array}{r} 5\ 0\ '2 \\ \_\ \_\ \_ \\ \hline 4\ 6\ 3 \end{array}$$

Now, 6 single caramels from 9 leaves 3.

Likewise, we cannot obtain 6 packages from 0 packages. We break open a carton to have:

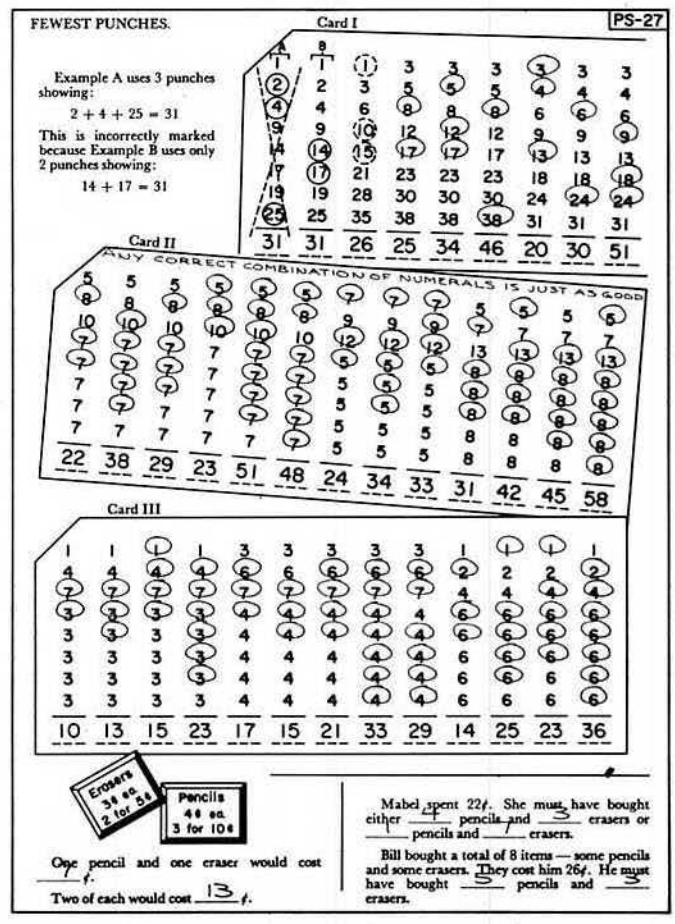
$$\begin{array}{r} 4\ '0\ '2 \\ \quad\ 6 \\ \hline 4\ 6\ 3 \end{array}$$

Now, taking 1 package from 7 leaves 6.

$$\begin{array}{r} 4\ '0\ '2 \\ \quad 1\ 6 \\ \hline 4\ 6\ 3 \end{array}$$

Practice sheet 27 contains more *Punch Card Arithmetic.* Part of its fascination for pupils may well be that it can be done without writing numbers. Division can be indicated in this way:

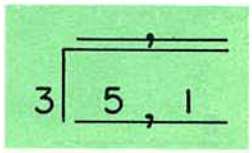
45.... (9) (9) (9) (9) (9) 9 9 9 9 9
56.... (7) (7) (7) (7) (7) (7) (7) (7) 7 7
36.... (6) (6) (6) (6) (6) (6) 6 6 6 6
35.... (5) (5) (5) (5) (5) (5) (5) 5 5 5
27.... (3) (3) (3) (3) (3) (3) (3) (3) (3) 3


FEWEST PUNCHES.
PS-27
Card I
Example A uses 3 punches showing:
2 + 4 + 25 = 31
This is incorrectly marked because Example B uses only 2 punches showing:
14 + 17 = 31
31 31 26 25 34 46 20 30 51
Card II
ANY CORRECT COMBINATION OF NUMERALS IS JUST AS GOOD
22 38 29 23 51 48 24 34 33 31 42 45 58
Card III
10 13 15 23 17 15 21 33 29 14 25 23 36
Erasers 3¢ ea. 2 for 5¢
Pencils 4¢ ea. 3 for 10¢
One pencil and one eraser would cost 7 ¢.
Two of each would cost 13 ¢.
Mabel spent 22¢. She must have bought either 1 pencils and 3 erasers or 4 pencils and 1 erasers.
Bill bought a total of 8 items — some pencils and some erasers. They cost him 26¢. He must have bought 5 pencils and 3 erasers.

"I was very slow in division examples at the Caramel Company," Tom told his friends. "But we invented a method that helped a lot.

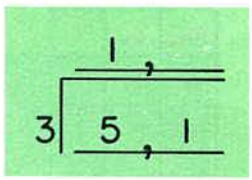
"I'll explain. We will begin with an easy example:

A_1



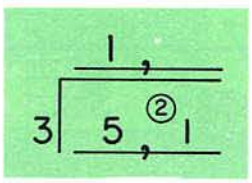
"I think — I've got 5 packages and 1 caramel to divide among 3 people. I can give each person a full package. So, I write:

A_2



. . . but I have 2 full packages left over. So I write ② in the next space — next to the 1 in this example.

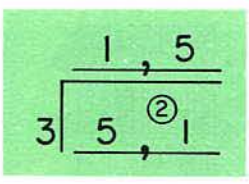
A_3



"I use the circle to help me remember that it is 2 *packages* I have not yet divided.

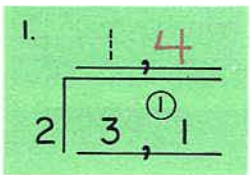
"Since there are 14 caramels in 2 packages and there is another caramel — 15 in all — I give 5 to each of the three people and write:

A_4

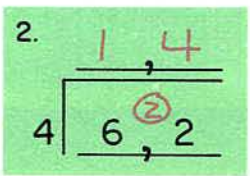


Here are two more examples. Please try to complete them yourself using Tom's method.

1.

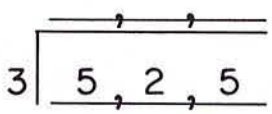


2.



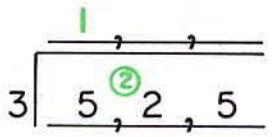
Harder examples follow a similar pattern. Here is one:

B_1



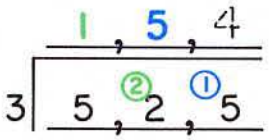
"I think — 5 boxes, 2 packages, and five caramels to divide among 3 people. I begin by giving each a box — and note that I have 2 of the 5 boxes left.

B_2



"I think of the ② boxes left over as 14 packages. These added to the two I started with make 16 packages. I can give 5 packages to each of the 3 people — and have 1 package left. So I write:

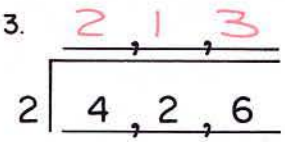
B_3



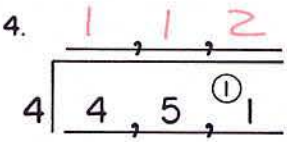
"Finally, ① package plus 5 caramels is 12 caramels — 4 to each person."

Here are more examples for you to try, using Tom's method.

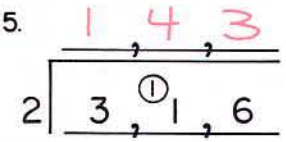
3.



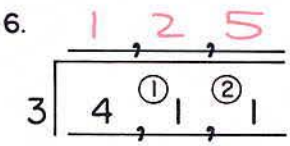
4.



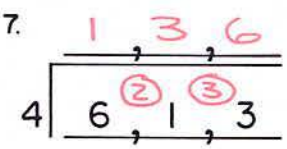
5.



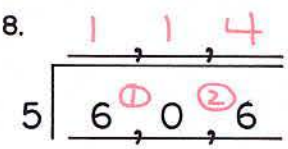
6.



7.



8.



28-29 Developing a kind of "short division" in this system

Tasks: Pupils study the division algorithm Tom uses and use it to find quotients in exercises 1 and 2. They then study the algorithm used on the right and use it to find quotients in exercises 3-8.

Purpose: To explore the base-seven system of numeration while perfecting whole-number computational skills in ordinary base-ten notation.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: Take your time with pupil pages 28 and 29. Here we work out the short division algorithm for Caramel Company arithmetic. The algorithm is the same as its counterpart in our decimal system of numeration. Talk through several exercises on the pupil page in Caramel Company language.

Exercise 1: An order for 3 packages and 1 separate caramel is to be divided into 2 shipments of the same size. We could start the division by putting 1 package in each of two piles we are building. The package left must be opened. Its 7 caramels are put with the 1 we had to make 8. Then we put 4 in each pile. So we find we have 1 package and 4 caramels in each shipment.

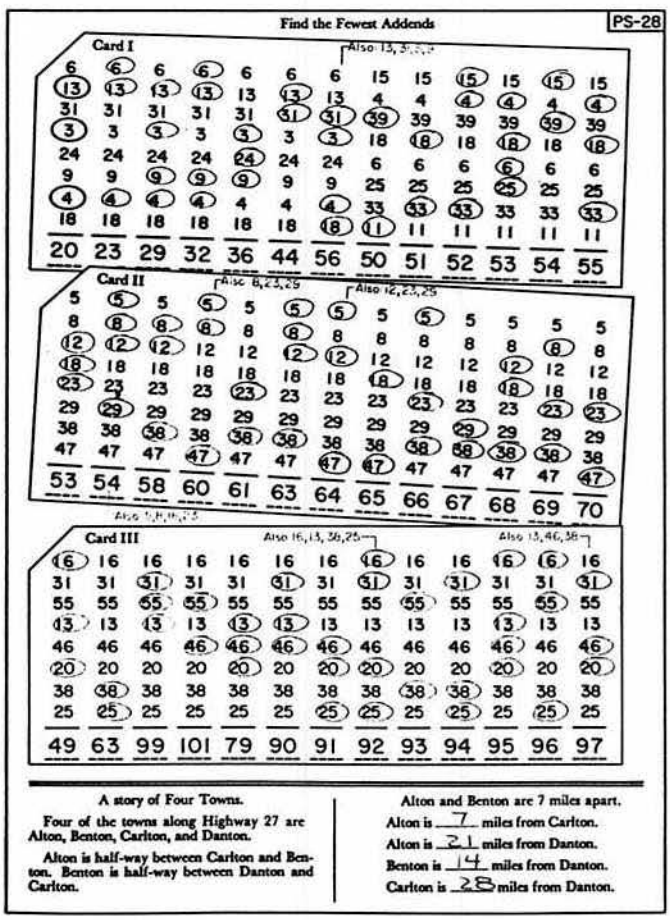
Exercise 2: We want to divide 6 packages and 2 caramels into 4 shipments of equal size. We begin by putting 1 full package in each pile. The two packages we have left must be opened and the 14 caramels put together with the 2 we already have to make a total of 16 caramels. We distribute these equally among the four piles—4 in each of the 4 piles. Thus, each of our 4 shipments should contain 1 package and 4 caramels.

We can make the same kind of argument later when we wish to divide \$7.38 into 3 parts of the same value. We put 2 dollars in each of 3 piles, leaving \$1.38 to be distributed. We change the \$1.00 into 10 dimes and put these with the 3 dimes of the 38¢, making 13 dimes. We put 4 in each pile, and change the remaining dime into 10 pennies to put with the 8 pennies yet to be distributed. The 18 pennies can be distributed 6 to a pile. We find that each pile contains 2 dollars, 4 dimes, and 6 pennies, or \$2.46

When one understands the arguments developed above he sees that the marks we make simply help us keep track of the activity. No rules, as such, need to be introduced; we are simply recording our progress as we distribute things among the required number of piles.

The tendency in recent years to skip over the short division algorithm is unfortunate. The algorithm and the method of step-by-step record-keeping helps pupils understand the use of approximations in more sophisticated algorithms.

Practice sheet 28 requires a lot of column addition. You may wish to extend this type of activity by having pupils call out their favorite numbers. Suppose that they call out: 7, 31, 15, 24, 12, 11, 20, 8. Ask them to find combinations whose sum is 19, 22, 26, etc. With numbers chosen at randon, they may find more than one way to express a number as the sum of the given numbers. For example: $35 = 15 + 20 = 24 + 11$. Have the class look for a collection of numbers for which no two sums are the same (a characteristic of many earlier examples of *Punch Card Arithmetic*).


Find the Fewest Addends
PS-28
Card I
20 23 29 32 36 44 56 50 51 52 53 54 55
Card II
53 54 58 60 61 63 64 65 66 67 68 69 70
Card III
49 63 99 101 79 90 91 92 93 94 95 96 97
A story of Four Towns.
Four of the towns along Highway 27 are Alton, Benton, Carlton, and Danton.
Alton is half-way between Carlton and Benton. Benton is half-way between Danton and Carlton.
Alton and Benton are 7 miles apart.
Alton is 7 miles from Carlton.
Alton is 21 miles from Danton.
Benton is 14 miles from Danton.
Carlton is 28 miles from Danton.

Tom used his methods in problems involving cartons as well as boxes, packages, and separate caramels.

Tom explained, "I can take care of the cartons first, then the boxes, then the packages, and finally — the loose caramels."

A
$$\begin{array}{r} 3024 \\ 2\overline{)6051} \end{array}$$

B
$$\begin{array}{r} 1506 \\ 2\overline{)3315} \end{array}$$

C
$$\begin{array}{r} 2054 \\ 3\overline{)6225} \end{array}$$

D
$$\begin{array}{r} 2435 \\ 2\overline{)5203} \end{array}$$

E
$$\begin{array}{r} 0513 \\ 3\overline{)2142} \end{array}$$

F
$$\begin{array}{r} 0242 \\ 5\overline{)1603} \end{array}$$

G
$$\begin{array}{r} 1006 \\ 4\overline{)4033} \end{array}$$

H
$$\begin{array}{r} 0312 \\ 6\overline{)2505} \end{array}$$

I
$$\begin{array}{r} 1155 \\ 4\overline{)5016} \end{array}$$

J
$$\begin{array}{r} 2036 \\ 3\overline{)6144} \end{array}$$

K
$$\begin{array}{r} 1206 \\ 5\overline{)6342} \end{array}$$

L
$$\begin{array}{r} 0224 \\ 3\overline{)1005} \end{array}$$

Here is a test Tom made up using Caramel Company arithmetic.

a
$$\begin{array}{r} 32 \\ +15 \\ \hline 50 \end{array}$$

b
$$\begin{array}{r} 60 \\ -33 \\ \hline 24 \end{array}$$

c
$$\begin{array}{r} 13 \\ \times 3 \\ \hline 42 \end{array}$$

d
$$\begin{array}{r} 204 \\ +064 \\ \hline 301 \end{array}$$

e
$$\begin{array}{r} 3165 \\ +1512 \\ \hline 5010 \end{array}$$

f
$$\begin{array}{r} 12 \\ \times 5 \\ \hline 63 \end{array}$$

g
$$\begin{array}{r} 46 \\ -23 \\ \hline 23 \end{array}$$

h
$$\begin{array}{r} 26 \\ \times 2 \\ \hline 55 \end{array}$$

i
$$\begin{array}{r} 304 \\ -162 \\ \hline 112 \end{array}$$

j
$$\begin{array}{r} 2002 \\ +2665 \\ \hline 5000 \end{array}$$

k
$$\begin{array}{r} 16 \\ 2\overline{)35} \end{array}$$

l
$$\begin{array}{r} 136 \\ 3\overline{)444} \end{array}$$

m
$$\begin{array}{r} 121 \\ 5\overline{)635} \end{array}$$

n
$$\begin{array}{r} 1252 \\ 4\overline{)5401} \end{array}$$

Tom Gets Shocking News!..... (continued on page 30)

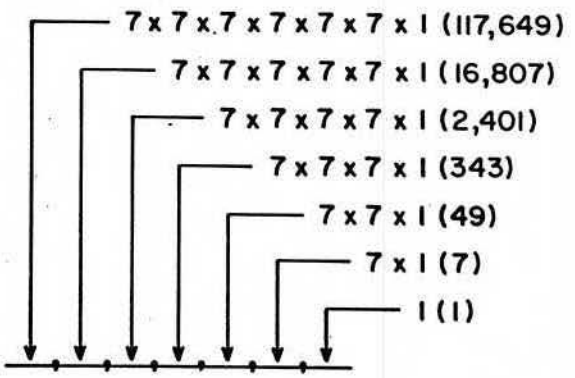
Tasks: *In exercises A-L pupils use Tom's short division algorithm to find the quotients. In exercises* a-u *pupils test their ability to carry out computations using base-seven notation.*

Purpose: To explore the base-seven system of numeration while perfecting whole-number computational skills in ordinary base-ten notation.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: On this final page of Caramel Company arithmetic using base-seven notation, we extend the short division algorithm to four-digit dividends. Nothing new is introduced—nor would anything new be encountered if we used larger dividends (as long as each new container was made up of 7 of the next smaller containers).

Some pupils may wish to expand the base-seven packing scheme:



Some may wish to investigate the system that would have been developed if the factory had packaged 5 caramels in a box, 5 boxes in a carton, 5 cartons in a crate, 5 crates in a skid, etc.

Caramel Company arithmetic in base seven, most of the *Punch Card Arithmetic* problems, and any of the extensions mentioned above are examples of the same basic notion that we use in our base-ten system—the place-value idea, which, in the decimal system, makes a digit in any place worth 10 times as much as the same digit in the place to its immediate right.

You can probably anticipate the development in Caramel Company arithmetic that starts on pupil page 30.

A new pattern of numbers for *Punch Card Arithmetic* appears in exercises 1-34 on practice sheet 29. This pattern makes it possible to indicate any number by one and only one combination of punches. Can you extend the pattern from the examples below?

A

4	2	1

B

9	3	1
9	3	1

C

16	4	1
16	4	1
16	4	1

D

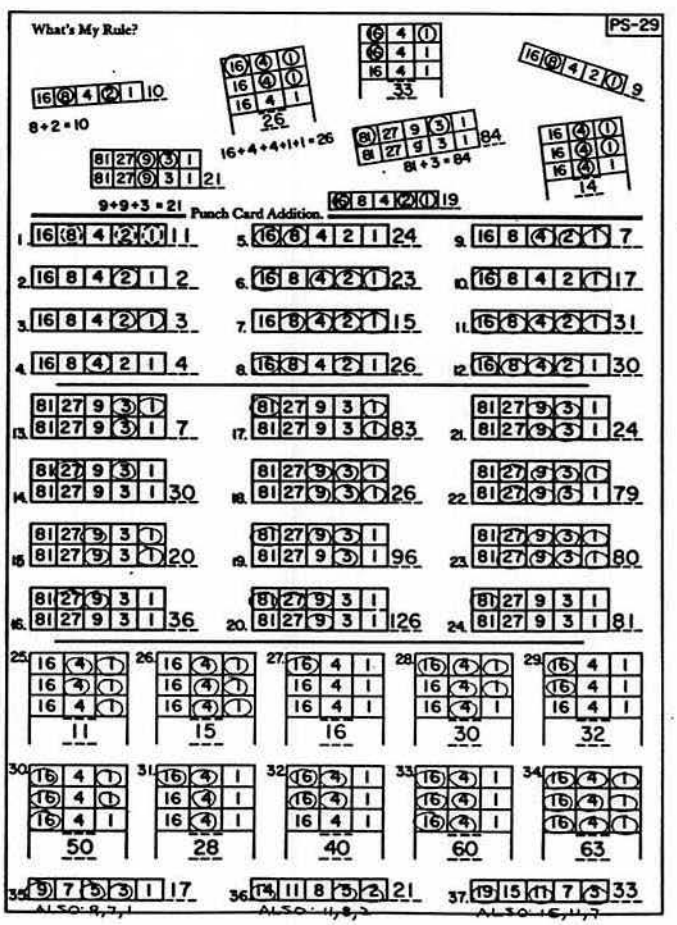
25	5	1
25	5	1
25	5	1
25	5	1

E

36	6	1
36	6	1
36	6	1
36	6	1
36	6	1

F

49	7	1
49	7	1
49	7	1
49	7	1
49	7	1
49	7	1



Tom came home from vacation one summer and found the following notice on the bulletin board:

> **IMPORTANT**
>
> Caramel Company has decided to pack 10 caramels in a package, 10 packages in a box, 10 boxes in a carton.
>
> The President.

Tom said, "Oh well, there goes my special Caramel Company arithmetic! Now I'll have to start all over."

First, Tom made a chart showing how many caramels there would be in each package, box, and carton.

1 package ..... 10 caramels

1 box ..... 10 packages
100 caramels

1 carton ..... 10 boxes
100 packages
1000 caramels

Tom experimented with his old sketch-short-hand:

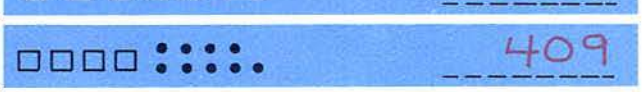
•..... a caramel —.... a package
□.... a box ⧉..... a carton

No. of caramels

a. 47

b. 1111

c. 2360

d.  409

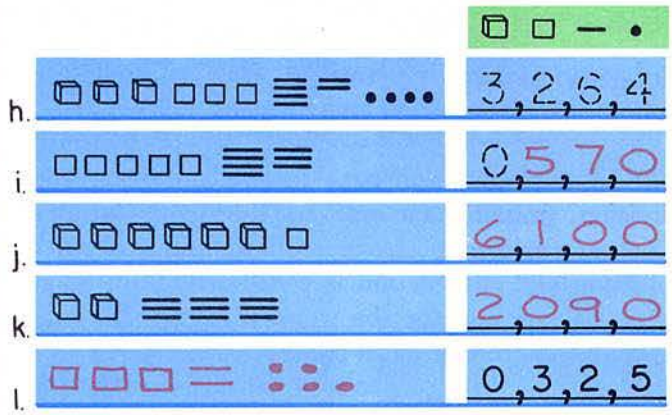
e. 3182

f. 98

g. 326

A smile appeared on Tom's face. "Ahah! I'm beginning to like this new packing scheme.

"There is one big trouble," Tom noted. "Sometimes I need to sketch as many as 9 containers of the same kind or 9 separate caramels." So, he quickly changed from his sketches to a quicker method — much as he had before.



"Next I'll try Caramel Company addition with this new packing arrangement.

m.
1,0,0,6
+ 3,0,0,7
4,0,1,3

n.
1,8,6,9
+ 0,6,0,1
2,4,7,0

o.
4,0,9,7
+ 3,0,0,5
7,1,0,2

p.
3,2,7,6
+ 1,7,8,3
5,0,5,9

"That wasn't hard. Let's try some examples of subtraction."

q.
5,2,6,4
− 1,1,0,8
4,1,5,6

r.
4,0,2,3
− 0,7,1,7
3,3,0,6

Try some examples of your own.

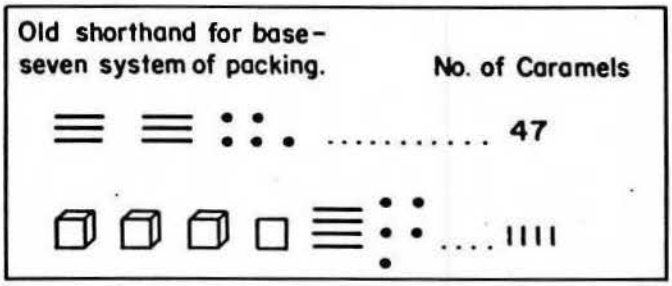
30 Changing the system to one based on multiples of 1, 10, 100, and 1,000 - a kind of decimal system

Tasks: *In exercises* a-f *pupils translate Tom's new shorthand notation into ordinary decimal notation. In exercise* g *pupils translate ordinary decimal notation into Tom's new shorthand notation. In exercises* i-l *they translate Tom's shorthand notation into a new Caramel Company notation. In exercises* m-r *they determine and record the indicated sums and differences.*

Purpose: To perfect whole-number computational skills while exploring decimal numeration.

Unifying Ideas: Numeration; addition and subtraction.

The Lesson: What an inconsiderate thing to do to Tom—to change the system of packaging after he had worked out such an effective shorthand and such efficient methods for adding, subtracting, multiplying, and dividing. He must start over again. But Tom soon finds that this new packaging system has certain advantages. Under the old packaging system, there was little obvious connection between his base-seven record and the ordinary base-ten record.



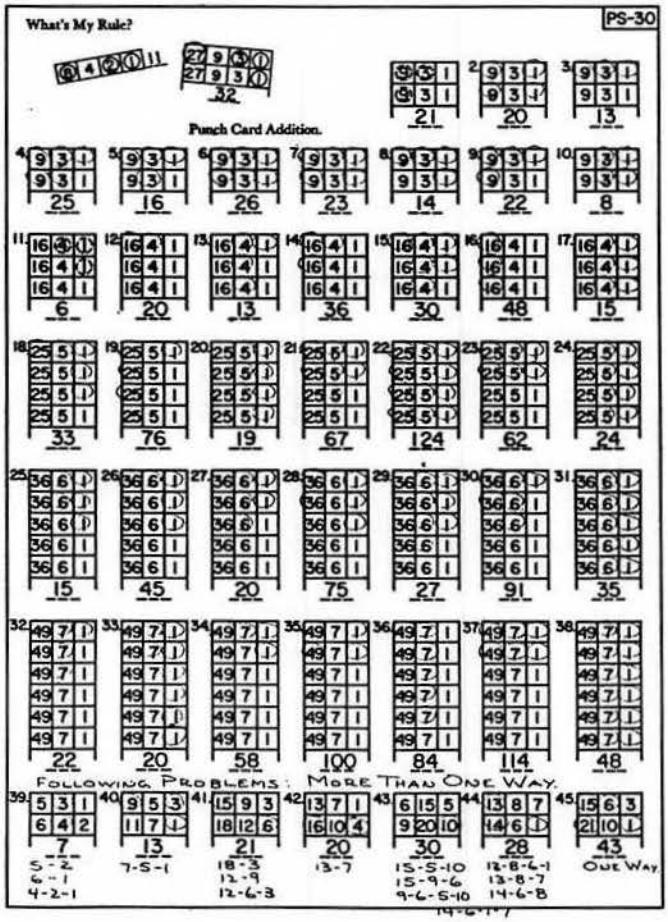
A quick look at the shorthand for the new base-ten system of packing reveals the number of caramels in the order. Likewise, the way we write numbers reveals at a glance the way the caramels are packed in the new system. This is true because the new packing system has the same scheme that is built into our decimal system of writing numbers.

In this new system, Tom finds that no single shipment will ever have more than 9 separate caramels or more than 9 containers of any one kind. The reason is that as soon as 10 or more caramels or containers are needed they are packed in one of the next larger containers. This parallels the fact that there were never more than 6 separate caramels nor more than 6 containers of any one kind in the base-seven system of packing.

Examples of *Punch Card* addition on practice sheet 30 explore other systems of numeration:

Base five, with place values of 25, 5, 1
Base six, with place values of 36, 6, 1
Base seven, with place values of 49, 7, 1

The last system, base seven, is another way of looking at the shorthand records for the original Caramel Company system of packaging.



Here are some more examples of Caramel Company Subtraction.

1
$$\begin{array}{r} 0,4,2,3 \\ -\ 0,2,0,7 \\ \hline 0,2,1,6 \end{array}$$

2
$$\begin{array}{r} 7,4,6,0 \\ -\ 0,7,5,8 \\ \hline 6,7,0,2 \end{array}$$

3
$$\begin{array}{r} 8,3,2,9 \\ -\ 4,1,5,3 \\ \hline 4,1,7,6 \end{array}$$

4
$$\begin{array}{r} 0,1,0,0 \\ -\ 0,0,3,6 \\ \hline 0,0,6,4 \end{array}$$

5
$$\begin{array}{r} 2,3,5,9 \\ -\ 0,3,7,3 \\ \hline 1,9,8,6 \end{array}$$

6
$$\begin{array}{r} 6,3,0,0 \\ -\ 3,1,5,3 \\ \hline 3,1,4,7 \end{array}$$

7
$$\begin{array}{r} 7,0,0,0 \\ -\ 4,1,2,5 \\ \hline 2,8,7,5 \end{array}$$

8
$$\begin{array}{r} 9,3,1,6 \\ -\ 4,7,5,8 \\ \hline 4,5,5,8 \end{array}$$

Here are some examples of Caramel Company Multiplication.

9
$$\begin{array}{r} 2,3,0,1 \\ \times\ 3 \\ \hline 6,9,0,3 \end{array}$$

10
$$\begin{array}{r} 0,9,0,7 \\ \times\ 2 \\ \hline 1,8,1,4 \end{array}$$

11
$$\begin{array}{r} 1,5,2,5 \\ \times\ 3 \\ \hline 4,5,7,5 \end{array}$$

12
$$\begin{array}{r} 2,1,3,2 \\ \times\ 4 \\ \hline 8,5,2,8 \end{array}$$

13
$$\begin{array}{r} 0,6,4,2 \\ \times\ 5 \\ \hline 3,2,1,0 \end{array}$$

14
$$\begin{array}{r} 0,4,3,1 \\ \times\ 8 \\ \hline 3,4,4,8 \end{array}$$

15
$$\begin{array}{r} 1,4,6,3 \\ \times\ 6 \\ \hline 8,7,7,8 \end{array}$$

16
$$\begin{array}{r} 0,8,4,3 \\ \times\ 7 \\ \hline 5,9,0,1 \end{array}$$

17
$$\begin{array}{r} 0,2,3,1 \\ \times\ 7 \\ \hline 1,6,1,7 \end{array}$$

18
$$\begin{array}{r} 0,2,3,1 \\ \times\ 10 \\ \hline 2,3,1,0 \end{array}$$

19
$$\begin{array}{r} 0,2,3,1 \\ \times\ 17 \\ \hline 3,9,2,7 \end{array}$$

20
$$\begin{array}{r} 0,2,3,1 \\ \times\ 27 \\ \hline 6,2,3,7 \end{array}$$

Watch the Signs!

21
$$\begin{array}{r} 3,7,2,5 \\ +\ 4,8,4,7 \\ \hline 8,5,7,2 \end{array}$$

22
$$\begin{array}{r} 4,3,6,9 \\ -\ 1,8,8,7 \\ \hline 2,4,8,2 \end{array}$$

23
$$\begin{array}{r} 1,7,1,3 \\ \times\ 4 \\ \hline 6,8,5,2 \end{array}$$

24
$$\begin{array}{r} 7,3,4,6 \\ -\ 5,2,9,3 \\ \hline 2,0,5,3 \end{array}$$

25
$$\begin{array}{r} 1,7,0,0 \\ \times\ 3 \\ \hline 5,1,0,0 \end{array}$$

26
$$\begin{array}{r} 3,7,4,6 \\ +\ 6,1,5,5 \\ \hline 9,9,0,1 \end{array}$$

27
$$\begin{array}{r} 9,4,1,6 \\ -\ 6,7,8,9 \\ \hline 2,6,2,7 \end{array}$$

28
$$\begin{array}{r} 5,2,8,4 \\ -\ 0,9,0,9 \\ \hline 4,3,7,5 \end{array}$$

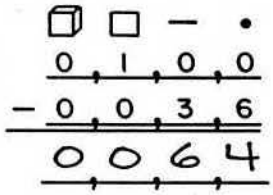
Tasks: *In exercises 1-8 pupils carry out the necessary computations to determine the missing addends (differences). In exercises 9-20 pupils compute the missing products. In exercises 21-28 they compute the missing sums, addends, or factors (quotients) as required.*

Purpose: To perfect computational skills for whole numbers in a variation of decimal notation.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

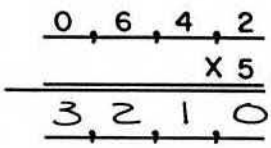
The Lesson: On pupil page 31 multiplication and division (missing factors) in the new Caramel Company arithmetic are introduced, and work with addition and subtraction is continued. Pupil page 31 is background for the work with the short division algorithm on pupil pages 32, 33, and 35.

It may be necessary to talk through some of the exercises. Consider Exercise 4:



From a box of caramels we are to remove 3 packages and 6 separate caramels. Obviously, the box must be opened, so we will have 10 packages. One of these packages must be opened, so we now have 9 packages and 10 separate caramels. If we remove 3 of these 9 packages and 6 of the 10 separate caramels, 6 packages and 4 caramels remain.

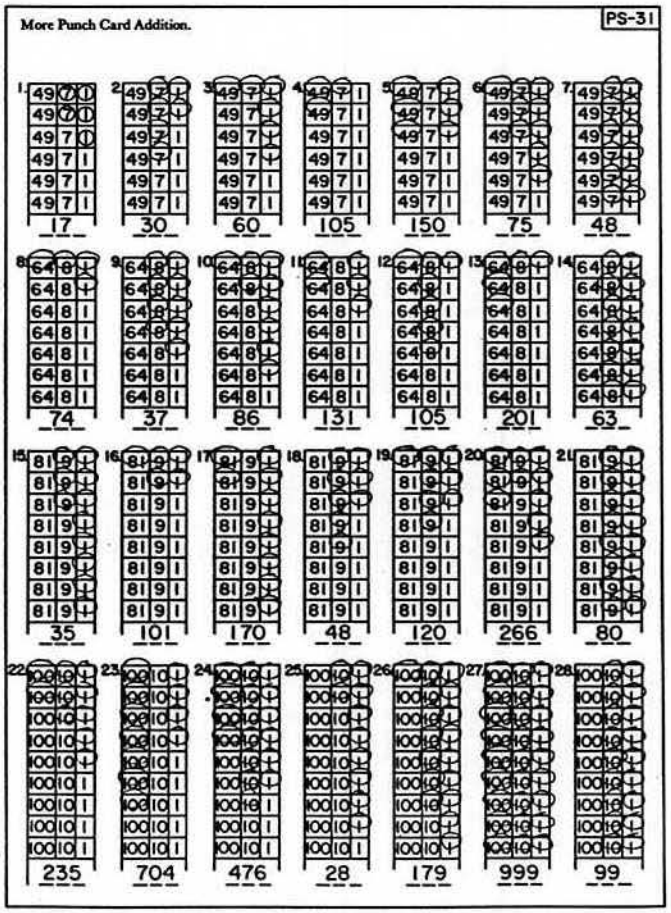
In exercise 13 we multiply an order of 6 boxes, 4 packages, and 2 separate caramels by 5.



Multiplying the 2 separate caramels by 5 makes 10—which can be put into a package, leaving no separate caramels. Multiplying the 4 packages by 5 makes 20 packages in addition to the one just made up from separate caramels. After the 21 packages are made into 2 boxes, 1 package is left. Multiplying 6 boxes by 5 makes 30 more boxes, which, with the 2 we already have, gives us 32. The 32 can be packed in 3 cartons with 2 boxes left over. Thus, we have finally: 3 cartons, 2 boxes, 1 package, and 0 separate caramels.

The pupil's work on practice sheet 31 becomes more difficult as he goes on—until he reaches the base-ten examples. Then, suddenly, the problems become too simple, since that *Punch Card* system and our decimal system of numeration have the same basic pattern.

Notice that there is only one way to punch out any given number, and any given number (if not too large) can be punched out on any of the cards. This can be seen clearly in exercise 27, where there are 9 hundreds, 9 tens, and 9 ones or units. Any number can be punched if it is less than 10,000.



Caramel Company Division.

"We adapted our Caramel Company way of doing division to the new packing system," Tom explained.

"Here is an example.

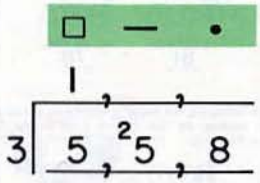
"Suppose I have 5 boxes, 5 packages, and 8 loose caramels and I want to divide them into 3 piles with the same number of caramels in each pile.

5 boxes (500 caramels) | 5 packages (50 caramels) | 8 loose caramels (8 caramels)

"That's a total of 558 caramels."

Step One.

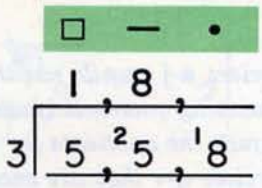
"First, I can put one of the boxes in each new pile . . . leaving 2 boxes I must change into packages. I record this first step this way:



"The two boxes can be opened and become 20 packages to add to the 5 I already have . . . a total of 25. By writing the '2' as I show it, it serves a double purpose. It shows how many boxes I take the packages out of . . . and writing it to the left of the '5' is a quick way of noting 25 packages."

Step Two.

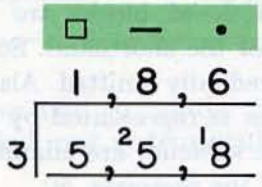
"Of the 25 packages, I can put 8 in each of the three piles, leaving 1 package undivided. I record that step this way."



Step Three.

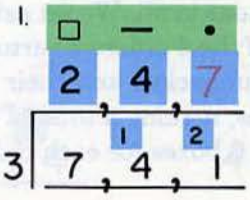
"There isn't much left to do. I open the package of 10 caramels, add them to the 8 loose ones . . . for a total of 18 loose caramels.

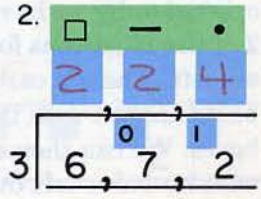
"I put 6 caramels in each pile and I'm done!

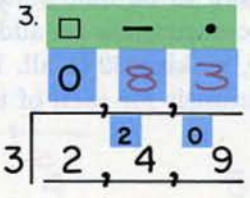


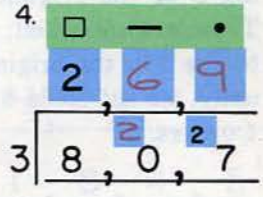
"I now have 1 box, 8 packages, and 6 loose caramels in each pile.

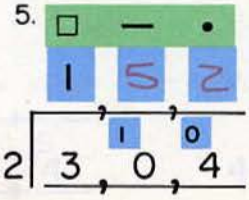
"Here are some examples with little blue blocks that indicate my work. I'll leave some of the blue blocks blank . . . for you to fill in according to my system."

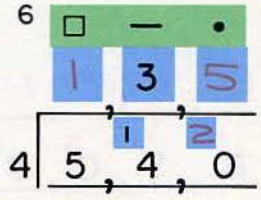
1.


2.


3.


4.


5.


6


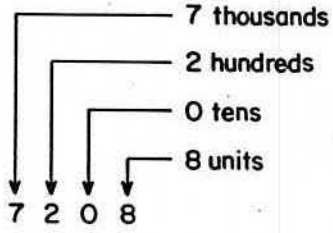
32-33 Developing a system of "short division" in our decimal system

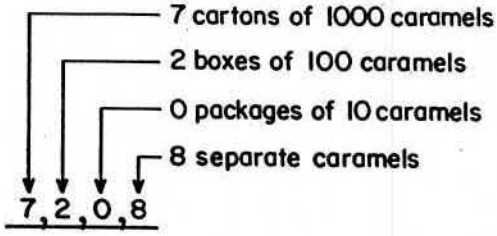
Tasks: Pupils study the three-step presentation of a short division algorithm and apply the algorithm to compute the quotients in exercises 1-6.

Purpose: To explore a short division algorithm for whole numbers in decimal notation.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: Pupils who already have a facility with a short division algorithm will immediately see through the Caramel Company system. After all, the company's new packaging scheme has much in common with our decimal or base-ten system of numeration.

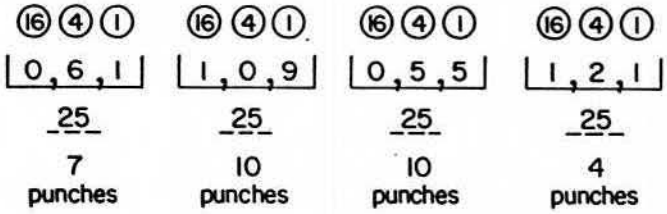




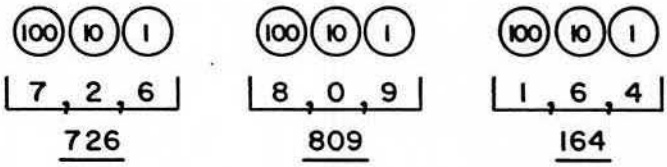
If the pupils already fully understand why the usual method of short division works, they need not spend much time on pupil pages 32 through 35. If they know the mechanics but not the rationale behind the process, these pages will help reveal the "whys."

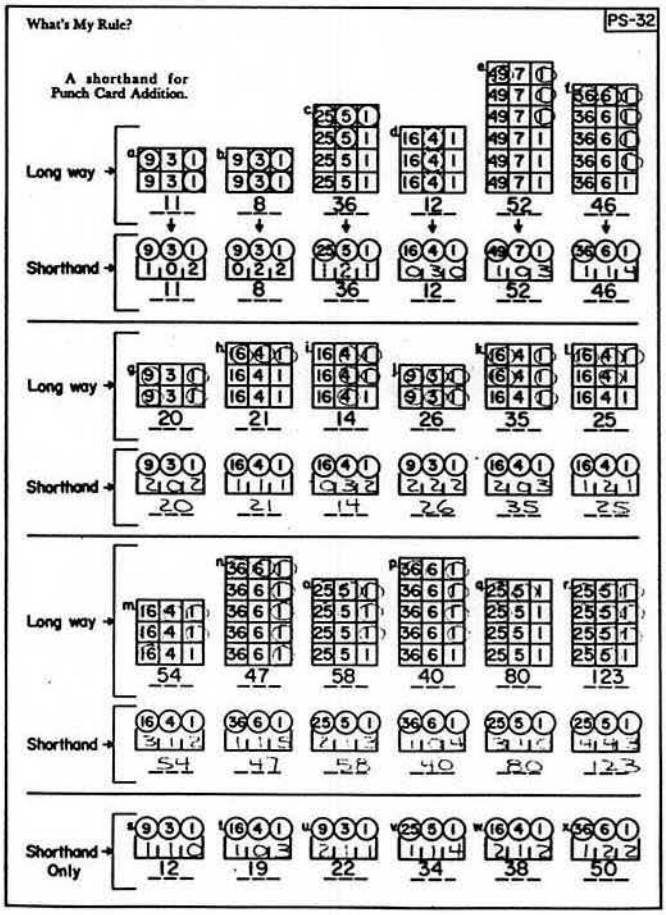
Pupil page 32 "talks through" a problem in terms of caramels and the new system of packaging. Talk through in similar fashion as many of the six additional exercises as you feel your pupils need help with.

Practice sheet 32 introduces a shorthand method for recording numbers in *Punch Card Arithmetic.* It leads to elimination of the lists of numbers available for punching. Pupils need to be reminded that they are expected to indicate numbers with the *fewest punches.* While the arithmetic in all the following examples is correct, only the last is acceptable since it consists of the fewest punches.



The numbers written in the circles are chosen in such a way that there will always be a single arrangement of fewest punches. Our decimal system is based on such an arrangement:





More examples of:

Caramel Company Division.

(We will omit the marks that indicate boxes, packages, and loose caramels.)

a. 165 / 5 | 825 — 3, 2

b. 043 / 6 | 258 — 2, 1

c. 103 / 8 | 824 — 0, 2

d. 035 / 7 | 245 — 2, 3

"We'll omit the large blue blocks."

e. 032 / 9 | 288 — 2, 1

f. 469 / 2 | 938 — 1, 1

g. 068 / 3 | 204 — 2, 2

h. 143 / 5 | 715 — 2, 1

i. 025 / 4 | 100 — 1, 2

j. 105 / 6 | 630 — 0, 3

"Of course," Tom pointed out, "this system can handle problems including cartons of caramels.

And, when we had to handle 'skids' with 10 cartons per skid, our system could help us find results."

k. 1425 / 3 | 4275 — 1, 0, 1

l. 1427 / 5 | 7135 — 2, 1, 3

m. 12618 / 4 | 50472 — (a skid) — 1, 2, 0, 3

We can begin omitting the large blue blocks.

n. 0468 / 5 | 2340 — 2, 3, 4

o. 1419 / 6 | 8514 — 2, 1, 5

p. 15282 / 4 | 61128 — 2, 1, 3, 0

Finally, we can begin omitting the small blue blocks.

q. 0362 / 7 | 2534 — 2, 4, 1

r. 0403 / 8 | 3224 — 3, 0, 2

s. 14369 / 5 | 71845 — 2, 1, 3, 4

t. 1042 / 9 | 9378 — 0, 3, 1

u. 1549 / 6 | 9294 — 3, 2, 5

v. 24519 / 4 | 98076 — 1, 2, 0, 3

Notes on Pupil Page 33

Tasks: *In exercises* a-j *pupils use the clues provided to compute the missing quotients (factors). In exercises* k-p *pupils compute the quotients for larger dividends. Finally, in exercises* q-v *they are encouraged to eliminate several recording steps and to carry out certain computations mentally to determine the missing quotients.*

Purpose: To perfect computational skills with a short division algorithm for whole numbers.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: Colored blocks are clearly not an essential part of the shorthand. So, on pupil page 33, they are gradually omitted. Also, we remember which container is represented by each place in a numeral so the sketches are eliminated. Let's talk through one of the exercises, *m*:

$$4\overline{)5,0,4,7,2}$$

What happens if an order for 5 skids, 0 cartons, 4 boxes, 7 packages, and 2 loose caramels is divided into 4 orders with the same number of caramels in each order?

We begin by setting aside 1 skid for each of the 4 orders. The remaining skid is then unpacked to form 10 cartons. Since there were 0 cartons in the original order we have 10 cartons in all. We set aside 2 of the 10 cartons for each of the 4 orders;2 cartons are left. These 2 cartons are unpacked and their 20 boxes are put with the 4 boxes, making a total of 24 boxes. We can then set aside 6 boxes for each order with no boxes left over.

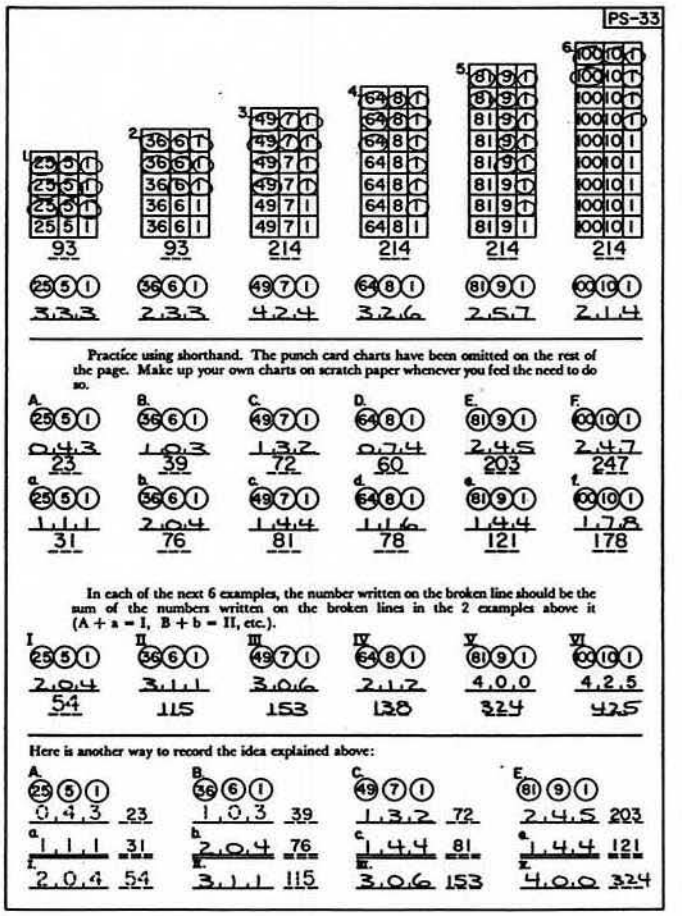
Next, we set aside 1 package for each order, leaving 3 of the original order yet to be distributed. These are unpacked; the 30 loose caramels are added to the 2 in the original order, making 32 in all. Finally, we set aside 8 loose caramels for each of the 4 orders.

Thus, each new order is composed of 1 skid, 2 cartons, 6 boxes, 1 package, and 8 loose caramels.

$$\begin{array}{r} 1,2,6,1,8 \\ \hline 4\overline{)5,0,4,7,2} \end{array}$$

Do not rush into a discussion of the wider or more general use of this algorithm. On pupil pages 34 and 35 we explore its application to computations involving amounts of money.

On practice sheet 33 we further explore the shorthand for *Punch Card Arithmetic* and find that problems such as 6, F, *f*, and VI are curiously easy.



Betty's Discovery.

"Tom!" Betty exploded. "Your new caramel packing scheme is a lot like the way we write down amounts of money."

"A dime is like a package of 10 pennies.

"A dollar is like a box of 10 dimes.

"A ten-dollar bill is like a carton of 10 dollars."

Betty's Guess.

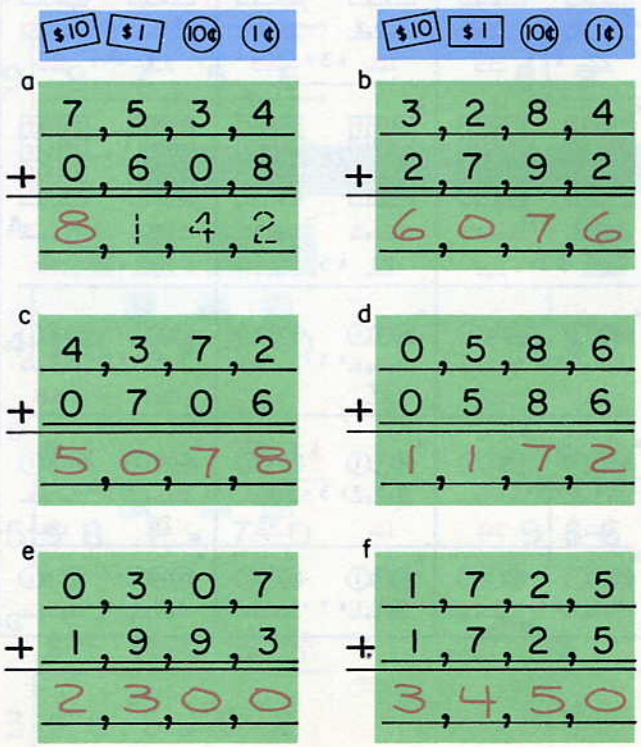
"And, I'll bet you can use your Caramel Company methods to add and subtract amounts of money and to multiply and divide them by numbers.

"Let's try!"

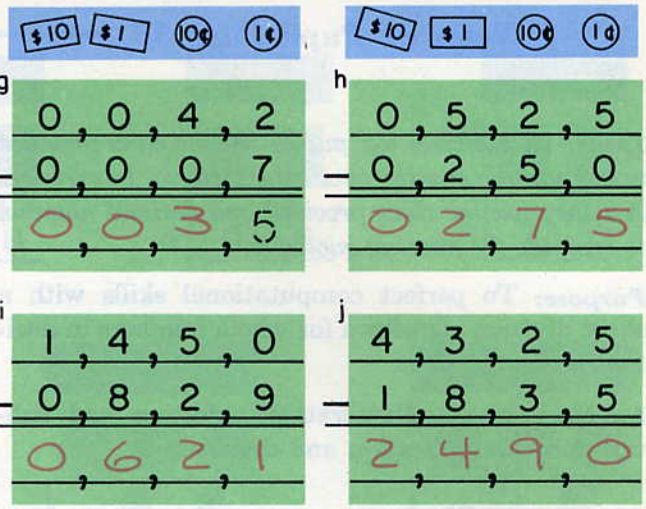
Everyone helped to find out whether Betty's guess was a good one.

Here are some of the examples they tried. Please fill in the blanks.

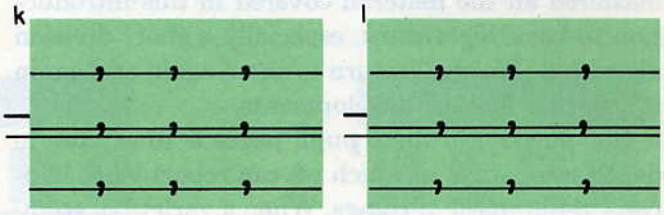
ADDITION.



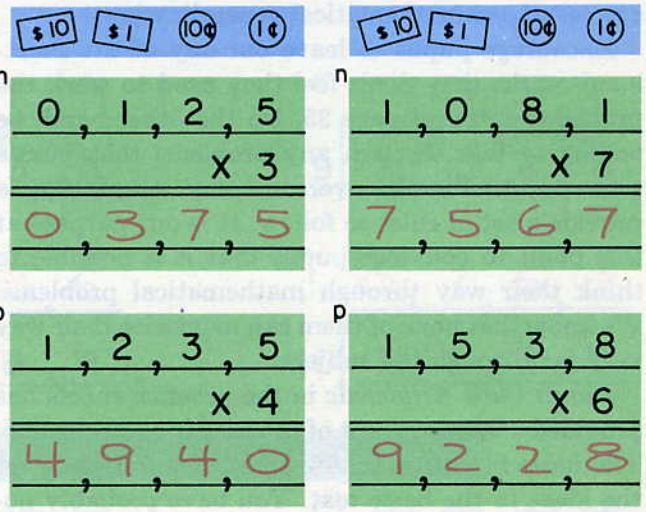
SUBTRACTION.



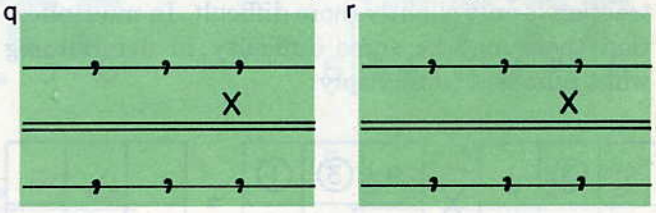
You make up two examples.



MULTIPLICATION.



You make up two examples.



"I thought I was right!" Betty was excited.

"And I think that kind of division you showed us will work with amounts of money."

Tasks: *In exercises* a-f *pupils compute the missing sums. In exercises* g-l *pupils compute the missing addends. Finally, in exercises* m-r *they compute the missing products.*

Purpose: To perfect computational skills with whole numbers in decimal notation.

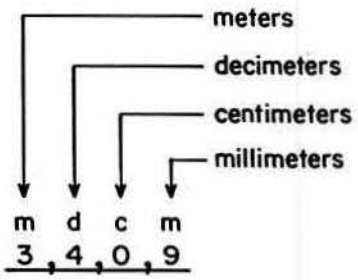
Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: You might well consider at this point a class discussion of the metric system of linear measurement. In this system, each unit is 10 times the next smaller unit, or each unit is one tenth of the next larger unit. That is precisely the scheme of the coins and bills considered on pupil page 34.

10 millimeters = 1 centimeter
10 centimeters = 1 decimeter
10 decimeters = 1 meter

Examine a meterstick. It's a little less than 40 inches long (39.37 inches). It is usually marked off to show all the units mentioned above.

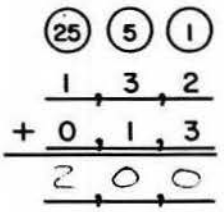
We can use the Caramel Company system and our money system to do Metric System arithmetic. Only the names change; the ideas do not. Suppose that we agree to use the millimeter as the basic unit.



As in other systems with the same basic structure, we use no more than 9 of any unit to represent a length. If we have 10 of any unit, we record it as 1 of the next larger unit.

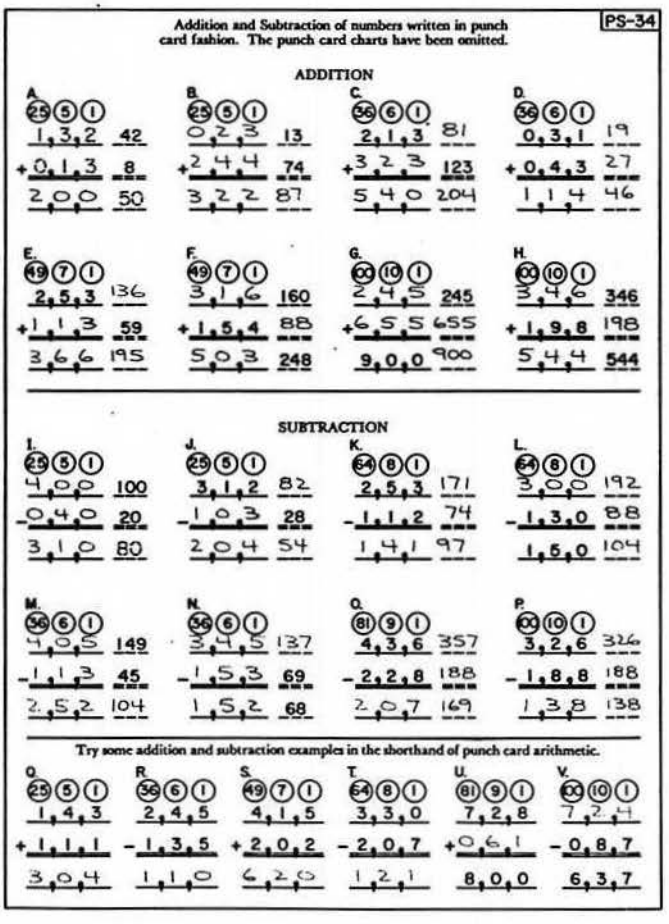
In this system, we can add and subtract magnitudes just as we did quantities of caramels and amounts of money. And we can multiply magnitudes by a number just as we did quantities of caramels and amounts of money.

On practice sheet 34 Punch Card shorthand is extended to addition and subtraction. One way of thinking about exercises A and B is to think of quarters, nickels, and pennies.

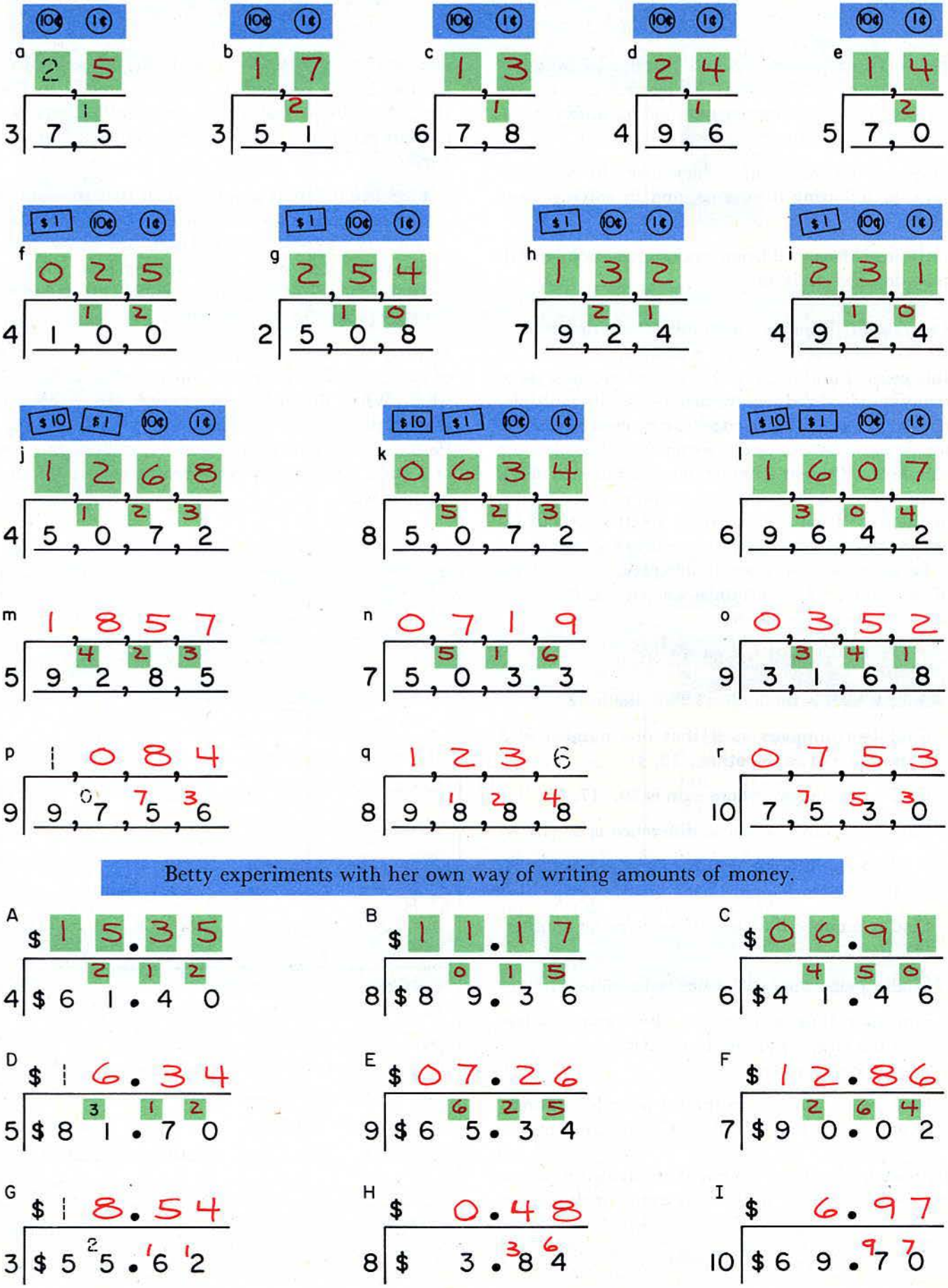


Two pennies and 3 pennies added together are 5 pennies, which we exchange for 1 nickel. This nickel added to the sum of 3 nickels and 1 nickel gives 5 nickels, which we exchange for 1 quarter. This quarter added to the other quarter gives a total of 2 quarters.

Base-six, base-seven, base-eight, or base-nine Caramel Company packaging systems can be used to think through other exercises.



DIVISION.



Betty experiments with her own way of writing amounts of money.

35 Developing a method of "short division" in handling amounts of money

Tasks: In exercises a-r *pupils use the clues provided to compute the missing quotients (factors). In exercises A-I they use the clues provided and decimal notation to compute the missing quotients.*

Purpose: To perfect computational skills with a short division algorithm for whole numbers in decimal notation.

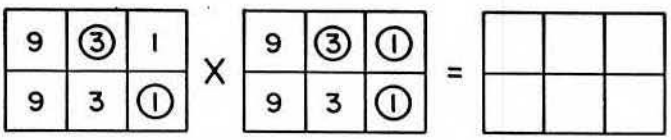
Unifying Ideas: Numeration; addition and subtraction; multiplication and division.

The Lesson: We do not expect all pupils to have mastered all the material covered in this introduction to basic algorithms, especially a short division algorithm. We shall return to them again and again to provide further developments.

Our purpose in these pupil pages is to explore in depth some ways in which we can report what happens in the world of things. When a variety of seemingly different activities have much in common we should strongly suspect that they are different examples of *one* mathematical generalization.

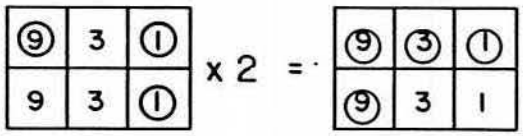
Encourage pupils to leave out any of the shorthand marks they don't feel they need to work the problems on pupil page 35. On the other hand, be willing to talk through any problems that puzzle some pupils. Finally, overcome any temptation to provide a set of rules to follow. It is our purpose at this point to convince pupils that it is possible to think their way through mathematical problems. We know that none of them can memorize their way very far through the subject.

Punch Card Arithmetic in the practice sheets has provided a large amount of practice in computation. The ideas in most of these sheets are variations of the ideas in the basic text. You have probably noticed that *Punch Card* addition is fairly easy. Subtraction is only slightly more difficult. In multiplication there may be some difficulty in determining what numbers to multiply:



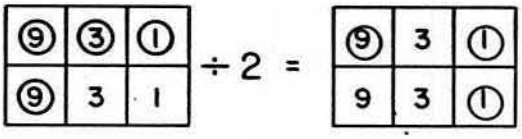
We could translate this into $4 \times 5 = 20$ and punch out the third card.

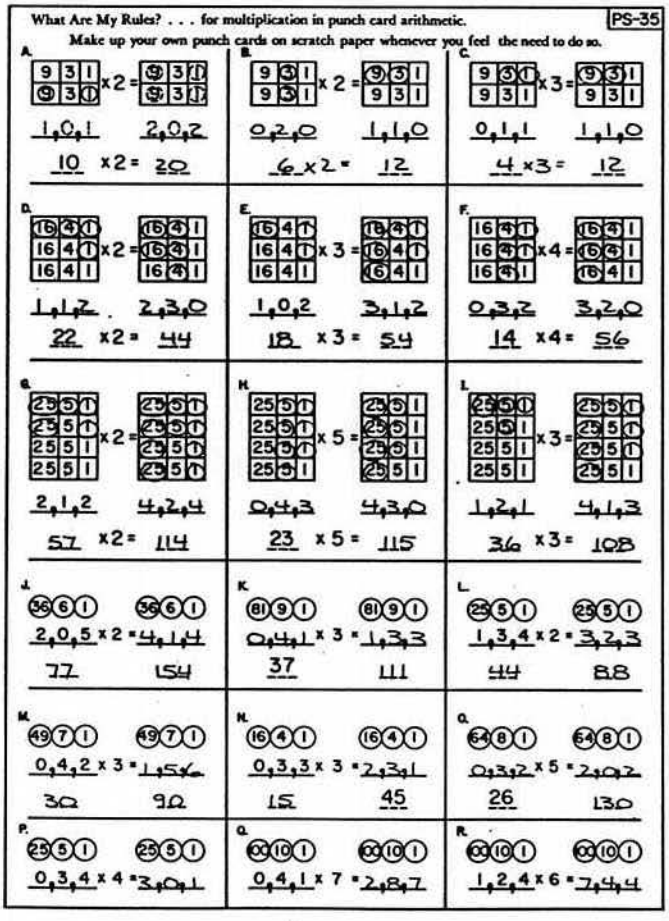
However, a more interesting type of problem is:



This tells us to punch out twice as much in the answer card as is represented on the card we are given. That is: two 9's and four 1's. Since there are only two 1's on the answer card, we punch out one 3 and one 1.

You may wish to experiment with dividing by a number:





Following Directions.

Mark the correct answer. Mark the furthest from the correct answer.

45 +18 — (1) 27 (2) 63 (3) 60 (4) 13

(1) ○ (2) ● (3) ○ (4) ⊗ — 50 — The difference between the answers you marked.

1 84 +36 = 120

(1) 110
(2) 120
(3) 48
(4) 52

(1) ○ (2) ● (3) ⊗ (4) ○ — 72

2 73 −37 = 36

(1) 37
(2) 44
(3) 100
(4) 36

(1) ○ (2) ○ (3) ⊗ (4) ● — 64

3 $3.45 × 2 = $6.90

(1) $6.54
(2) $3.47
(3) $6.90
(4) $3.42

(1) ○ (2) ○ (3) ● (4) ⊗ — $3.48

4 134 + 578 + 47 = 759

(1) 759
(2) 849
(3) 659
(4) 769

(1) ● (2) ○ (3) ⊗ (4) ○ — 100

5 $17.42 − 12.84 = $4.58

(1) $30.26
(2) $ 5.42
(3) $ 4.58
(4) $29.42

(1) ⊗ (2) ○ (3) ● (4) ○ — $25.68

6 7 × 9 = 63

(1) 72
(2) 56
(3) 63
(4) 79

(1) ○ (2) ○ (3) ● (4) ⊗ — 16

Sale School Supplies

Notebooks 24¢ ea.
Pen and Pencil Sets 39¢ ea.
Lunch Boxes 53¢ ea.

7 How much did Martha pay for one of each?

(1) $1.16
(2) $1.17
(3) 63¢
(4) 96¢

(1) ● (2) ○ (3) ⊗ (4) ○ — 53¢

8 Bill bought 2 notebooks and a pen and pencil set. How much did he pay?

(1) 63¢
(2) 87¢
(3) $1.02
(4) 77¢

(1) ⊗ (2) ● (3) ○ (4) ○ — 24¢

9 Helen bought a lunch box. She gave the clerk a dollar bill. How much change should she get?

(1) 57¢
(2) 53¢
(3) $1.53
(4) 47¢

(1) ○ (2) ○ (3) ⊗ (4) ● — $1.06

10 Alice bought 4 pen and pencil sets. How much did she spend?

(1) $1.17
(2) $1.36
(3) $1.56
(4) $4.39

(1) ○ (2) ○ (3) ● (4) ⊗ — $2.83

11 Harry bought 2 of each article. He gave the clerk a five-dollar bill. How much change should he get?

(1) $2.32
(2) $3.84
(3) $2.68
(4) $2.00

(1) ○ (2) ⊗ (3) ● (4) ○ — $1.16

12 Cora bought 3 notebooks, 2 pen and pencil sets, and a lunch box. How much did she spend?

(1) $2.03
(2) $1.16
(3) $2.23
(4) $1.23

(1) ● (2) ⊗ (3) ○ (4) ○ — 87¢

36 Introducing a typical multiple-choice question-asking method similar to one used in many standardized achievement tests

Tasks: In each exercise pupils perform the indicated computations to determine the correct answer, the furthest from the correct answer, and the difference between these two numbers.

Purpose: To test pupil achievement in computation, in following directions, and in solving story problems.

Unifying Ideas: Addition and subtraction; multiplication and division.

The Lesson: Pupil pages 36 and 37 are designed to be used as tests if you feel a checkup is needed at this point. Pupil page 36 is prepared in the style of many standardized achievement tests, with multiple-choice questions. These pages can be used as a model for similar sheets of practice exercises. The plan is a simple one. Present a limited number of items in the form of a price list, sale sign (as on pupil page 36), or the like. Then ask questions about relationships among the prices, or lengths, or times, or numbers.

Perhaps the most general illustration of this kind of problem is a list of numbers suggested by pupils and written on the board.

17, 9, 4, 13, 30, 12, 34, 6

Ask: Which is the largest? The smallest?

Find two numbers such that one number is 3 times as large as the other. (12, 4)

Find two numbers whose sum is 30. (17, 13)

Find two numbers whose difference is 9. (13, 4)

Find three different numbers whose sum is 34. (9, 13, 12)

Find a number that is half as large as another. (17, 34)

Find a pair whose difference is 18. (30, 12)

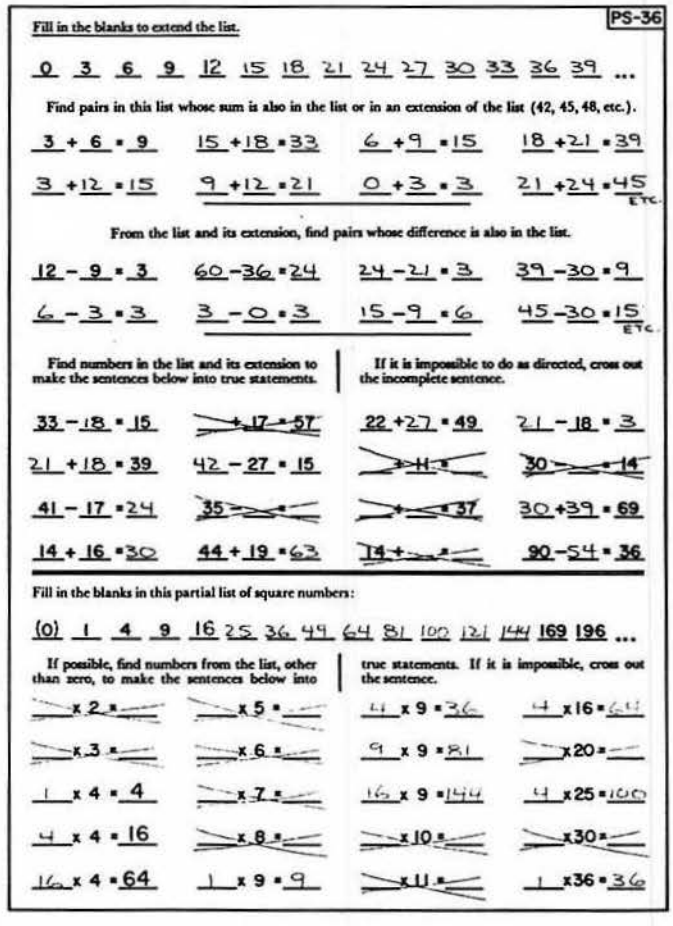
Find four different numbers whose sum is 2 less than the largest number in the list. (4+6+9+13=34−2)

How many numbers in the list are odd? (Three) Even? (Five) Prime? (Two) Composite? (Six)

With such a list the class might be divided into two or more groups for a game. A group makes up a problem and gives it in writing to the other group or groups. If the problem is solved correctly in less than one minute, the solvers get points; if not, the authors get points. Expect some complicated problems:

> Find five different numbers such that the sum of three of them is the same as the sum of the other two. (17+12+9=34+4. There are other combinations that meet the requirements, such as 9+4+34=17+30. A team might get twice as many points for two correct answers.)

Writing the problems focuses attention on clarity of expression and averts arguments about who said what. When differences of opinion arise, the discussion that follows will amost always be rewarding. Perhaps there should be a penalty, such as the loss of a turn, for questions that are written in an ambiguous way.



PS-36

Fill in the blanks to extend the list.

0 3 6 9 12 15 18 21 24 27 30 33 36 39 ...

Find pairs in this list whose sum is also in the list or in an extension of the list (42, 45, 48, etc.).

3 + 6 = 9 | 15 + 18 = 33 | 6 + 9 = 15 | 18 + 21 = 39

3 + 12 = 15 | 9 + 12 = 21 | 0 + 3 = 3 | 21 + 24 = 45 Etc.

From the list and its extension, find pairs whose difference is also in the list.

12 − 9 = 3 | 60 − 36 = 24 | 24 − 21 = 3 | 39 − 30 = 9

6 − 3 = 3 | 3 − 0 = 3 | 15 − 9 = 6 | 45 − 30 = 15 Etc.

Find numbers in the list and its extension to make the sentences below into true statements. | If it is impossible to do as directed, cross out the incomplete sentence.

33 − 18 = 15 | + 17 = 57 | 22 + 27 = 49 | 21 − 18 = 3

21 + 18 = 39 | 42 − 27 = 15 | + 11 = | 30 − = 14

41 − 17 = 24 | 35 − = | + = 37 | 30 + 39 = 69

14 + 16 = 30 | 44 + 19 = 63 | 14 + = | 90 − 54 = 36

Fill in the blanks in this partial list of square numbers:

(0) 1 4 9 16 25 36 49 64 81 100 121 144 169 196 ...

If possible, find numbers from the list, other than zero, to make the sentences below into true statements. If it is impossible, cross out the sentence.

x 2 = | x 5 = | 4 x 9 = 36 | 4 x 16 = 64

x 3 = | x 6 = | 9 x 9 = 81 | x 20 =

1 x 4 = 4 | x 7 = | 16 x 9 = 144 | 4 x 25 = 100

4 x 4 = 16 | x 8 = | x 10 = | x 30 =

16 x 4 = 64 | 1 x 9 = 9 | x 11 = | 1 x 36 = 36

Show the largest amount with an A in the green block; the next largest with a B; the next with a C; etc. Write the difference in each example.

I

3 yards, 2 feet, 7 inches ... 139 ... B

100 inches ... 100 ... C

12 feet ... 144 ... A

Difference between

A and B: 5 inches

B and C: 39 inches

A and C: 44 inches

II

15 dimes and 40 nickels ... 3.50 ... C

40 dimes and 230 pennies ... 6.30 ... A

57 dimes ... 5.70 ... B

Difference between

A and B: 60 ¢

B and C: 220 ¢

A and C: 280 ¢

III

37 hours ... 37 ... A

1 day and 5 hours ... 29 ... B

600 minutes ... 10 ... D

Half a day ... 12 ... C

Difference between

A and B: 8 hrs.

B and C: 17 hrs.

A and C: 25 hrs.

B and D: 19 hrs.

C and D: 2 hrs.

A and D: 27 hrs.

IV

An hour and a quarter ... 75 ... D

240 minutes ... 240 ... A

Three half-hours ... 90 ... C

1 hour and 90 minutes ... 150 ... B

Difference between

A and B: 90 MIN.

B and C: 60 MIN.

A and C: 150 MIN.

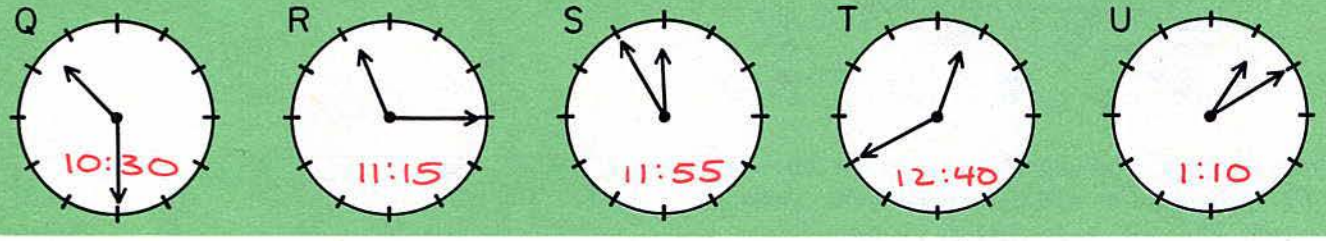
B and D: 75 MIN.

C and D: 15 MIN.

A and D: 165 MIN.

V

The game was to start at 1:15 P.M. Bill looked at the clock 5 different times before the game started.



How long was it before game time in each case?

Q : 165 min.

R : 120 min.

S : 80 min.

T : 35 min.

U : 5 min.

How long was it between the times he looked at the clock?

Q and R: 45 min.

R and S: 40 min.

S and T: 45 min.

T and U: 30 min.

S and U: 75 min.

R and T: 85 min.

Q and U: 160 min.

Q and S: 85 min.

37 Comparing measurements

Tasks: *In exercises I-IV pupils label the largest amount with an A, the next largest with a B, the next largest with a C, and, in III and IV, the smallest with a D. They then compute the indicated differences. In exercise V, pupils record the correct time indicated on the clock faces Q-U and compute the indicated differences.*

Purpose: To test pupil achievement in computation involving various systems of measurement and in following directions.

Unifying Ideas: Numeration; addition and subtraction; measurement.

The Lesson: Pupil page 37, a second checkup page, presents two variations of the exercises in the lower section of pupil page 36. Each variation presents information in a different way, and relationships among the given quantities are explored. Still another way of giving information is used on practice sheet 37.

Another way is simply to say to the pupils: Here is a situation. Think about it. Now, what can you say? For example, what can you say about:

Pencils—4¢ each Rulers—7¢ each Pads— 9¢ each
Limit: 3 each to a customer

The least interesting statements would be repetitions of the information given.

Pads cost 9¢ each. You can't buy more than 3 of them.

Pads are the most expensive items listed.

This is a bit more interesting.

The most a person could spend is 60¢.

An interesting observation.

You couldn't spend exactly 10¢.

If you spent 17¢, I would be sure you had bought 2 pencils and 1 pad.

If you spent 21¢, I would know you had bought 3 rulers or a pad and 3 pencils.

You can't spend 55¢.

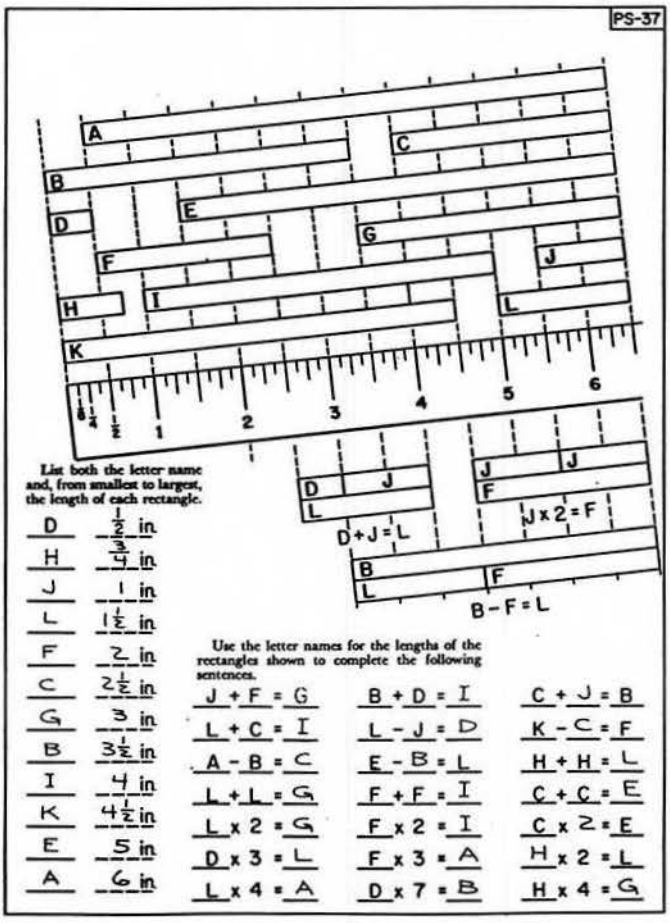
If you didn't buy any pads and spent 33¢, then you must have bought all the pencils and rulers you could buy.

There is no amount of money you can spend in three different ways.

This is far from obvious.

If the limit were lowered to 2 of each to a customer, then no amount of money could be spent in two different ways. All I would need to know is how much you spent and I could tell you exactly what you bought.

Thus very little information leads to countless questions, many of which require considerable thought. Further, thinking about such a situation results in lots of practice with the basic combinations of addition and multiplication. Such probing of inner relationships is the very stuff of mathematics. Thinking about such problems helps children understand more fully how arithmetic is applied to the real world about them.



More about Arithmetic of Things and Numbers (cont. from page 23)

Some classmates asked Ben and Ruth about multiplication and division in the "Arithmetic of Things and Numbers."

Jerry asked, "What would you do if your Mother gave you this problem?"

3 dogs x 2 dogs = _____

or

4 nickels x 5 dimes = _____

"Mother never gave us problems like those," Ruth told the class.

"Ben and I couldn't have made any sense out of such questions.

"In the arithmetic Mother taught us, we never multiplied things by things; rather, we multiplied things by numbers . . . and we divided things by numbers. We would have problems such as:

A 3 dogs x 2 = 6 dogs

B 4 nickels x 5 = 20 nickels

C 6 dogs ÷ 2 = 3 dogs

D 20 nickels ÷ 5 = 4 nickels

E $1.25 x 2 = $ 2.50

F 8 in. x 3 = 2 feet

G 7 in. x 4 = 2 ft., 4 in.

H 2 hrs. ÷ 3 = 40 minutes

I 1 day ÷ 6 = 4 hours

"But Ben and I considered such problems too easy. We would often answer the question asked — and several more at the same time.

"For example, Mother might say:

7 stars x 4 = 28 stars

"We would prepare lists like this. You can fill in the blanks."

J

Numbers of stars

	1	3	4	7	10	17	
x 4	4	12	16	28	40	68	

or

K

Numbers of stars

	2	5	7	12	19	31	
x 4	8	20	28	48	76	124	

or

L

Numbers of stars

	8	9	10	18	19	20	
x 4	32	36	40	72	76	80	

"Or, Mother might say:

24¢ ÷ 3 = 8¢

"Here are some lists we might prepare. I'm leaving some blanks for you to fill in."

M

Numbers of cents

	12	24	36	60	96	156	
÷ 3	4	8	12	20	32	52	

N

Numbers of cents

	24	30	60	108	90	120	
÷ 3	8	10	20	36	30	40	

Tasks: In exercises A-I pupils carry out the indicated operations to make each sentence a true statement. In exercises J-N they use the clues provided to do the necessary computations to complete each table.

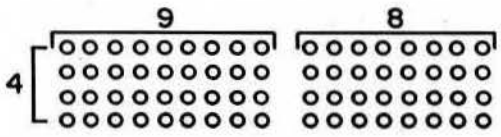
Purpose: To continue exploring the *Arithmetic of Things and Numbers* while perfecting computational skills for whole numbers.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; measurement.

The Lesson: Pupil pages 38 through 45 and the accompanying practice sheets do not, at first glance, seem related—except that both are concerned with the ideas of multiplication. However, the pupil pages and the practice sheets both focus attention on the distributive principle of multiplication over addition—the fundamental idea upon which algorithms of multiplication and division are based.

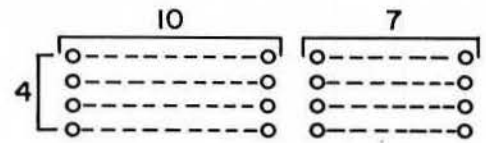
The pupil pages treat multiplication as a kind of repetitive addition. The practice sheets use representations of multiplication more closely related to Cartesian products—arrays, rows and columns, or whatever name you use for it.

We can use the distributive principle to find out how many people can be seated in 4 rows with 17 chairs in each row. If we do not remember that $17 \times 4 = 68$, we can break the problem into parts, use what we have memorized to do each part, and then bring the parts together again. We can draw a sketch to explain what we do:



From memory, we know the products of 9 by 4 and 8 by 4; they are 36 and 32, respectively. These we can add. We can report that 17×4 must be 68.

Someone more experienced in looking ahead to efficiency of bookkeeping may argue from a slightly different sketch:



Since $10 \times 4 = 40$ and $7 \times 4 = 28$, the result 68 follows from a simple addition.

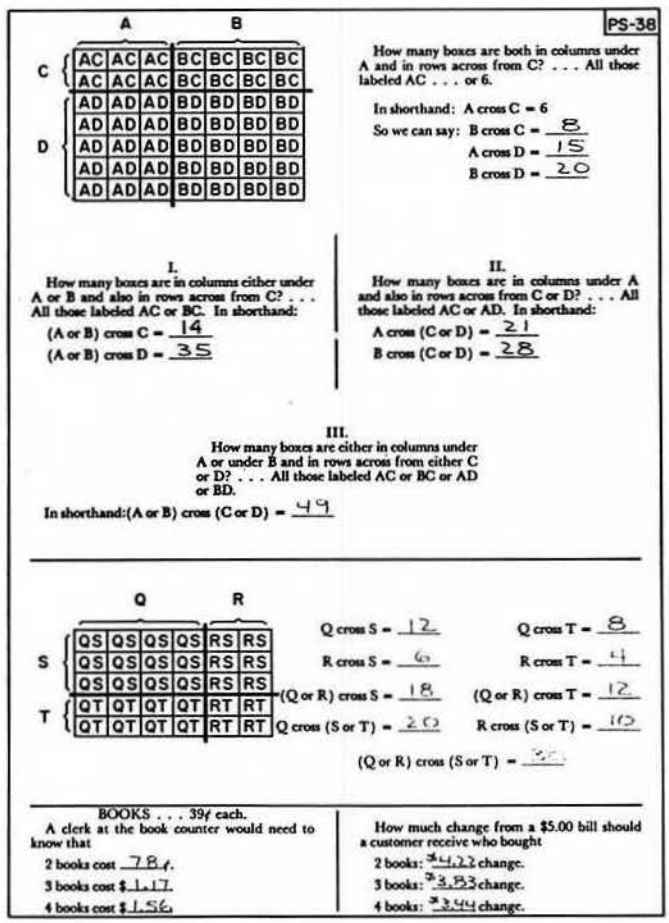
We might write these arguments:

$$17 \times 4 = (9+8) \times 4 = (9 \times 4) + (8 \times 4) = 36 + 32 = 68$$
$$17 \times 4 = (10+7) \times 4 = (10 \times 4) + (7 \times 4) = 40 + 28 = 68$$

The distributive principle is illustrated many times in the lists on Pupil Page 38. Notice the last three columns of chart J:

	7	10	17
x 4	28	40	68

On practice sheets 38, 39, and 40 the word *or* is used in the sense of set union. While pupils need not have this explained to them to understand how to work successfully on these pages, you may wish to study the technical meaning of *or* as it applies to sets in the Appendix of this manual.


PS-38
A
B
C
D
How many boxes are both in columns under A and in rows across from C? . . . All those labeled AC . . . or 6.
In shorthand: A cross C = 6
So we can say: B cross C = 8
A cross D = 15
B cross D = 20
I.
How many boxes are in columns either under A or B and also in rows across from C? . . . All those labeled AC or BC. In shorthand:
(A or B) cross C = 14
(A or B) cross D = 35
II.
How many boxes are in columns under A and also in rows across from C or D? . . . All those labeled AC or AD. In shorthand:
A cross (C or D) = 21
B cross (C or D) = 28
III.
How many boxes are either in columns under A or under B and in rows across from either C or D? . . . All those labeled AC or BC or AD or BD.
In shorthand: (A or B) cross (C or D) = 49
Q
R
S
T
Q cross S = 12
R cross S = 6
(Q or R) cross S = 18
Q cross (S or T) = 20
Q cross T = 8
R cross T = 4
(Q or R) cross T = 12
R cross (S or T) = 10
(Q or R) cross (S or T) = 30
BOOKS . . . 39¢ each.
A clerk at the book counter would need to know that
2 books cost 78¢.
3 books cost $1.17.
4 books cost $1.56.
How much change from a $5.00 bill should a customer receive who bought
2 books: $4.22 change.
3 books: $3.83 change.
4 books: $3.44 change.

"Sometimes Ben and I would give Mother answers that showed her we knew several ways to tackle hard problems. Here are several examples — with plenty of blanks for you."

A. 9 days x 7 = 63 days

Numbers of days

1.

	6	3	9
x 2	12	6	18
x 5	30	15	45
x 7	42	21	63

2.

	4	5	9
x 3	12	15	27
x 4	16	20	36
x 7	28	35	63

3.

	3	6	9
x 1	3	6	9
x 6	18	36	54
x 7	21	42	63

B. 16 pounds x 8 = 128 pounds

Numbers of pounds

4.

	9	7	16
x 3	27	21	48
x 5	45	35	80
x 8	72	56	128

5.

	11	5	16
x 4	44	20	64
x 4	44	20	64
x 8	88	40	128

6.

	6	10	16
x 1	6	10	16
x 7	42	70	112
x 8	48	80	128

C. 13 miles x 14 = 182 miles

Numbers of miles

7.

	6	7	13
x 8	48	56	104
x 6	36	42	78
x 14	84	98	182

8.

	9	4	13
x 7	63	28	91
x 7	63	28	91
x 14	126	56	182

9.

	3	10	13
x 4	12	40	52
x 10	30	100	130
x 14	42	140	182

D. 15¢ x 15 = 225 ¢

Numbers of cents

10.

	6	9	15
x 8	48	72	120
x 7	42	63	105
x 15	90	135	225

11.

	2	13	15
x 1	2	13	15
x 14	28	182	210
x 15	30	195	225

12.

	10	5	15
x 5	50	25	75
x 10	100	50	150
x 15	150	75	225

39-40 Introducing multiplication with two-digit multipliers

Tasks: In each exercise pupils use the clues provided by the design of the tables to devise useful shortcuts for computing the indicated products.

Purpose: To lay the groundwork for discovery of the distributive principle of multiplication over addition while perfecting computational skills with whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

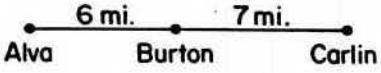
The Lesson: As pupils learn to use the ordinary algorithm for multiplication, they should often be asked how they might find a correct answer if they forgot the usual procedure. On pupil page 39, Ben and Ruth show that they can get the correct result in several different ways. Are there others? Certainly, and you might well ask your class to suggest several more and perhaps to support their suggestions with sketches.

Since 9 days is 1 week and 2 days, we might rewrite exercise A as:

$$\underline{\text{1 week and 2 days}} \times 7 = \_\_\_$$

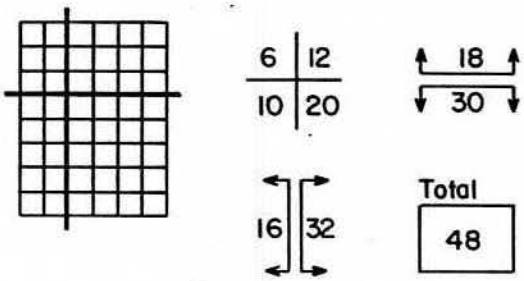
This product is 7 weeks and 14 days, or 49 days +14 days—a total of 63 days.

Exercise C can be thought of as a problem involving a delivery truck traveling to three towns along a highway.



The truck driver made 4 round trips from Alva to Carlin in one week. This means that he covered the distance from Alva to Burton 8 times for a total of 48 miles and the distance from Burton to Carlin 8 times for a total of 56 miles—104 miles in all. The next week, he made 3 round trips for a total of 78 miles. In the 7 round trips, he drove a total of 182 miles.

In each exercise on this pupil page and pupil page 40, pupils face a question whose answer they have not memorized. They break the question into parts that can be answered by recall and put the partial results together as an answer to the question. The basic idea behind this strategy is presented pictorially in accompanying practice sheets. If we want to know the number of tiles in a rectangular array, we can partition it, note the number of tiles in each part, and add these numbers. We might keep notes in any of these ways:

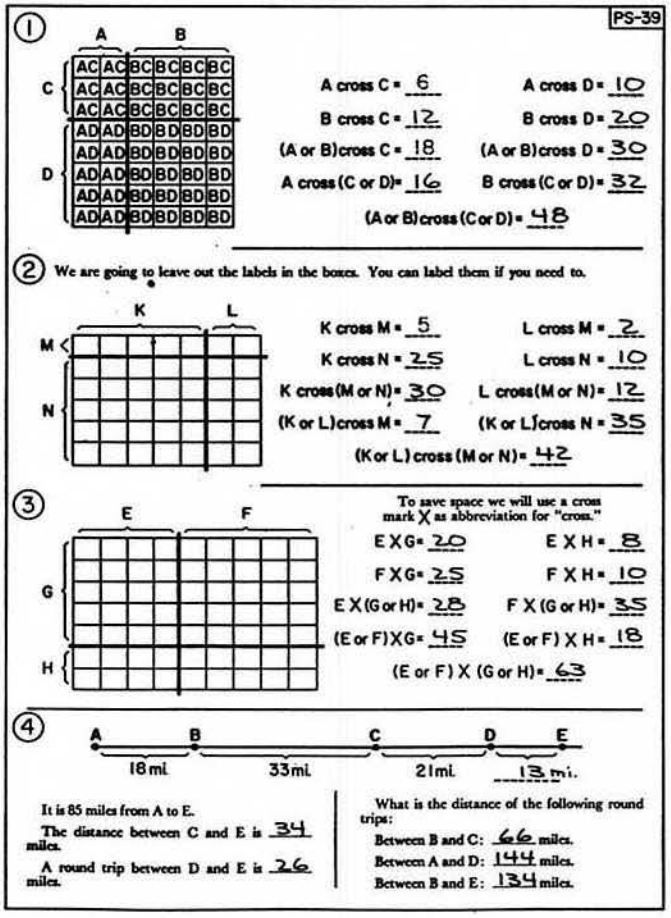


We might write out some of these arguments:

$$(2+4)\times 3 = (2\times 3)+(4\times 3) = 6+12 = 18$$
$$(2+4)\times 5 = (2\times 5)+(4\times 5) = 10+20 = 30$$
$$(2+4)\times 8 = (2\times 8)+(4\times 8) = 16+32 = 48$$

and:

$$2\times(3+5) = (2\times 3)+(2\times 5) = 6+10 = 16$$
$$4\times(3+5) = (4\times 3)+(4\times 5) = 12+20 = 32$$
$$6\times(3+5) = (6\times 3)+(6\times 5) = 18+30 = 48$$


PS-39
1
A
B
C
D
A cross C = 6
A cross D = 10
B cross C = 12
B cross D = 20
(A or B) cross C = 18
(A or B) cross D = 30
A cross (C or D) = 16
B cross (C or D) = 32
(A or B) cross (C or D) = 48
2
We are going to leave out the labels in the boxes. You can label them if you need to.
K
L
M
N
K cross M = 5
L cross M = 2
K cross N = 25
L cross N = 10
K cross (M or N) = 30
L cross (M or N) = 12
(K or L) cross M = 7
(K or L) cross N = 35
(K or L) cross (M or N) = 42
3
To save space we will use a cross mark X as abbreviation for "cross."
E
F
G
H
E X G = 20
E X H = 8
F X G = 25
F X H = 10
E X (G or H) = 28
F X (G or H) = 35
(E or F) X G = 45
(E or F) X H = 18
(E or F) X (G or H) = 63
4
A
B
C
D
E
18 mi.
33 mi.
21 mi.
13 mi.
It is 85 miles from A to E.
The distance between C and E is 34 miles.
A round trip between D and E is 26 miles.
What is the distance of the following round trips:
Between B and C: 66 miles.
Between A and D: 144 miles.
Between B and E: 134 miles.

"Mother was especially pleased when our list of results showed we were looking for unusual shortcuts."

A. 9 gallons x 8 = 72 gallons

Numbers of gallons

1.

	10	1	9
x 10	100	10	90
x 2	20	2	18
x 8	80	8	72

2.

	4	5	9
x 4	16	20	36
x 4	16	20	36
x 8	32	40	72

3.

	3	6	9
x 3	9	18	27
x 5	15	30	45
x 8	24	48	72

B. 15 feet x 7 = 105 feet

Numbers of feet

4.

	8	7	15
x 3	24	21	45
x 4	32	28	60
x 7	56	49	105

5.

	20	5	15
x 2	40	10	30
x 5	100	25	75
x 7	140	35	105

6.

	10	5	15
x 10	100	50	150
x 3	30	15	45
x 7	70	35	105

C. 19 pounds x 18 = 342 pounds

Numbers of pounds

7.

	11	8	19
x 9	99	72	171
x 9	99	72	171
x 18	198	144	342

8.

	20	1	19
x 20	400	20	380
x 2	40	2	38
x 18	360	18	342

9.

	10	9	19
x 10	100	90	190
x 8	80	72	152
x 18	180	162	342

D. $23 x 45 = $ 1035

Numbers of dollars

10.

	11	12	23
x 15	165	180	345
x 30	330	360	690
x 45	495	540	1035

11.

	20	3	23
x 50	1000	150	1150
x 5	100	15	115
x 45	900	135	1035

12.

	3	20	23
x 5	15	100	115
x 40	120	800	920
x 45	135	900	1035

Tasks: In each exercise pupils use the clues provided by the design of the table to devise useful shortcuts for computing the indicated products.

Purpose: To continue laying the groundwork for discovery of the distributive principle for multiplication over addition and to hint at the possibility of a distributive principle of multiplication over subtraction while perfecting computational skills with whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The distributive property is not limited to addition and multiplication. We find that for any numbers a, b, and c it is also true that

$$(a-b)\times c=(a\times c)-(b\times c)$$

(Since we have not introduced negative numbers, pupils would stipulate that b must not be greater than a.) For example:

$$\begin{aligned}(10-1)\times 7&=(10\times 7)-(1\times 7)\\&=70-7=63\end{aligned}$$

$$\begin{aligned}(25-3)\times 8&=(25\times 8)-(3\times 8)\\&=200-24=176\end{aligned}$$

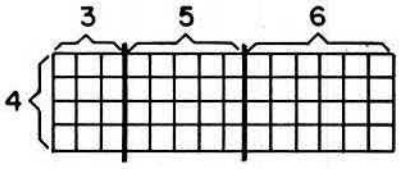
$$\begin{aligned}(40-2)\times 17&=(40\times 17)-(2\times 17)\\&=680-34=646\end{aligned}$$

These might be good ways of finding that

$$\begin{aligned}9\times 7&=63\\22\times 8&=176\\38\times 17&=646\end{aligned}$$

On pupil page 40 Ben and Ruth use the distributive principle of multiplication over subtraction to show that they can find several ways of doing any given problem.

On practice sheet 40 we explore another extension of the distributive principle. How many tiles in 4 rows with 14 in each row?



$$\begin{aligned}14\times 4&=(3+5+6)\times 4\\&=(3\times 4)+(5\times 4)+(6\times 4)\\&=12+20+24=56\end{aligned}$$

This solution is based on the observation that for any numbers a, b, c, and d

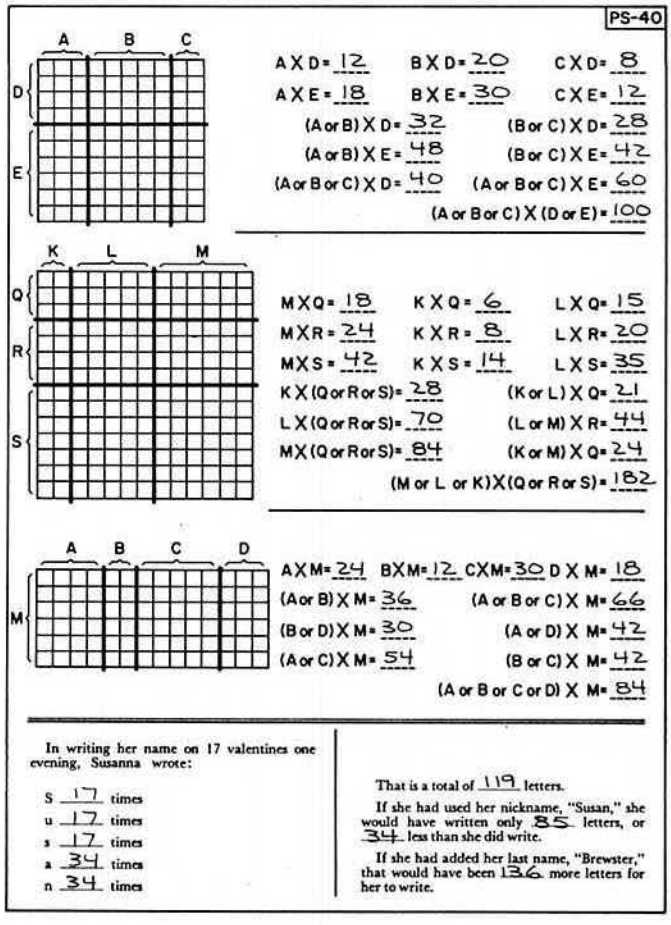
$$(a+b+c)\times d=(a\times d)+(b\times d)+(c\times d)$$

Although partitioning one factor into three addends is not really useful in the case of 14×4, it is useful in the case of 176×4. In fact, such a partition is used in the familiar algorithm:

$$(100+70+6)\times 4=?\quad\text{or}\quad\begin{array}{r}100+70+6\\\times\quad 4\\\hline\end{array}$$

Some pupils may develop enough understanding and confidence to look for such shortcuts as

$$\begin{aligned}176\times 4&=(200-25+1)\times 4\\&=800-100+4=704\end{aligned}$$


PS-40
A B C
D
E
A X D = 12
B X D = 20
C X D = 8
A X E = 18
B X E = 30
C X E = 12
(A or B) X D = 32
(B or C) X D = 28
(A or B) X E = 48
(B or C) X E = 42
(A or B or C) X D = 40
(A or B or C) X E = 60
(A or B or C) X (D or E) = 100
K L M
Q
R
S
M X Q = 18
K X Q = 6
L X Q = 15
M X R = 24
K X R = 8
L X R = 20
M X S = 42
K X S = 14
L X S = 35
K X (Q or R or S) = 28
(K or L) X Q = 21
L X (Q or R or S) = 70
(L or M) X R = 44
M X (Q or R or S) = 84
(K or M) X Q = 24
(M or L or K) X (Q or R or S) = 182
A B C D
M
A X M = 24
B X M = 12
C X M = 30
D X M = 18
(A or B) X M = 36
(A or B or C) X M = 66
(B or D) X M = 30
(A or D) X M = 42
(A or C) X M = 54
(B or C) X M = 42
(A or B or C or D) X M = 84
In writing her name on 17 valentines one evening, Susanna wrote:
S 17 times
u 17 times
s 17 times
a 34 times
n 34 times
That is a total of 119 letters.
If she had used her nickname, "Susan," she would have written only 85 letters, or 34 less than she did write.
If she had added her last name, "Brewster," that would have been 136 more letters for her to write.

"Mother taught us to do some of our work in our heads. She said that neatness was important and showed us how to write our work in little squares like this:"

A 26 pounds x 17 = 442 lbs.

	20	6	26
x 7	140	42	182
x 10	200	60	260
x 17	340	102	442

		2	6
	x	1	7
	1	8	2
	2	6	0
	4	4	2

B 39¢ x 41 = $ 1599

	30	9	39
x 1	30	9	39
x 40	1200	360	1560
x 41	1230	369	1599

		3	9
	x	4	1
		3	9
1	5	6	0
1	5	9	9

C 28 tiles x 53 = 1484 tiles

	20	8	28
x 50	1000	400	1400
x 3	60	24	84
x 53	1060	424	1484

		2	8
	x	5	3
		8	4
1	4	0	0
1	4	8	4

D 2 ft. and 8 in. x 39 = 1248 inches

Number of inches

	30	2	32
x 30	900	60	960
x 9	270	18	288
x 39	1170	78	1248

		3	2
	x	3	9
	2	8	8
	9	6	0
1	2	4	8

"For practice in neatness, she would leave some of the lines out. Soon, we didn't need any lines at all."

E

		3	6
	x	1	4
	1	4	4
	3	6	0
	5	0	4

F

	1	9
x	5	2
	3	8
9	5	0
9	8	8

G

	2	8
x	2	3
	8	4
5	6	0
6	4	4

H

	7	4
x	1	9
6	6	6
7	4	0
1	4	0	6

I

	5	1
x	4	7
	3	5	7
2	0	4	0
2	3	9	7

J

	9	2
x	2	9
	8	2	8
1	8	4	0
2	6	6	8

K

		6	6
	x	8	6
		3	9	6
5	2	8	0
5	6	7	6

L

	4	8
x	7	7
	3	3	6
3	3	6	0
3	6	9	6

M

	8	7
x	4	0
	0	0
3	4	8	0
3	4	8	0

N

	5	0
x	1	8
4	0	0
5	0	0
9	0	0

41 Introducing the algorithm for multiplication with two-digit multipliers

Tasks: *In exercises A-D pupils carry out the computations to complete each table. They also use the clues provided to record in the squares to the right of each table the partial products and sums of these products. In exercises E-N pupils compute and record partial products and sums to complete each multiplication example.*

Purpose: To apply the distributive principle to develop a multiplication algorithm for whole numbers.

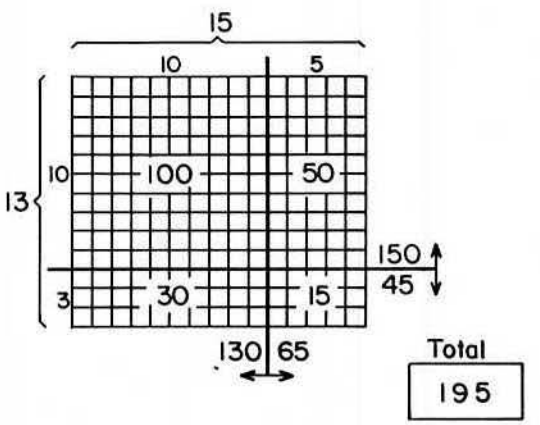
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The usual algorithm for multiplication is an efficient way to record results as we break a "hard" multiplication problem up into "easy" multiplication problems—and then add the partial results.

In exercises A, B, C, and D we combine the method used in the previous pupil pages with an improved bookkeeping method. Practice sheets 41 and 42 include an even more basic approach to the algorithm through the use of pictorial representations. Perhaps several examples should be done on the board showing all three approaches to the same problem including two uses of the algorithm:

	10	5	15
10	100	50	150
3	30	15	45
13	130	65	195

```
  15          13
 x13         x15
 ---         ---
 45          65
150         130
---         ---
195         195
```

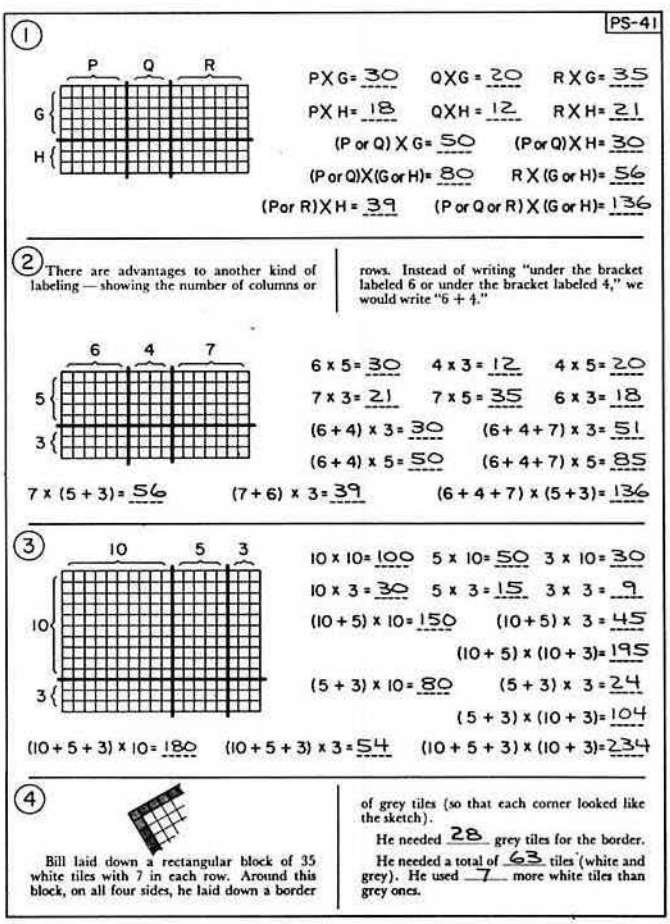


Since sketches showing all the tiles are unwieldy for even relatively small numbers, we can suggest the full sketch by an abbreviated form:

	10	5	
10	(100)	(50)	(150)
3	(30)	(15)	(45)
	(130)	(65)	195

Whenever a pupil stumbles in using a multiplication algorithm, he can go back to one of these sketches and reconstruct the steps of the algorithm that are illustrated by the sketches.

Exercises E through N provide an opportunity for discussion of the zero that occurs in the units place of the second partial product. Will a zero always turn up in that spot? Why? (Because we are always multiplying by a multiple of ten.) Let the pupils use their own judgment about omitting the zero. There is no reason to hurry this practice.


PS-41
1
P Q R
G
H
P X G = 30
Q X G = 20
R X G = 35
P X H = 18
Q X H = 12
R X H = 21
(P or Q) X G = 50
(P or Q) X H = 30
(P or Q) X (G or H) = 80
R X (G or H) = 56
(P or R) X H = 39
(P or Q or R) X (G or H) = 136
2
There are advantages to another kind of labeling — showing the number of columns or rows. Instead of writing "under the bracket labeled 6 or under the bracket labeled 4," we would write "6 + 4."
6 4 7
5
3
6 x 5 = 30
4 x 3 = 12
4 x 5 = 20
7 x 3 = 21
7 x 5 = 35
6 x 3 = 18
(6 + 4) x 3 = 30
(6 + 4 + 7) x 3 = 51
(6 + 4) x 5 = 50
(6 + 4 + 7) x 5 = 85
7 x (5 + 3) = 56
(7 + 6) x 3 = 39
(6 + 4 + 7) x (5 + 3) = 136
3
10 5 3
10
3
10 x 10 = 100
5 x 10 = 50
3 x 10 = 30
10 x 3 = 30
5 x 3 = 15
3 x 3 = 9
(10 + 5) x 10 = 150
(10 + 5) x 3 = 45
(10 + 5) x (10 + 3) = 195
(5 + 3) x 10 = 80
(5 + 3) x 3 = 24
(5 + 3) x (10 + 3) = 104
(10 + 5 + 3) x 10 = 180
(10 + 5 + 3) x 3 = 54
(10 + 5 + 3) x (10 + 3) = 234
4
Bill laid down a rectangular block of 35 white tiles with 7 in each row. Around this block, on all four sides, he laid down a border of grey tiles (so that each corner looked like the sketch).
He needed 28 grey tiles for the border.
He needed a total of 63 tiles (white and grey). He used 7 more white tiles than grey ones.

"When some of the numbers were larger, we had to expand our charts. We also did some mental arithmetic and recorded the results to the right of the charts."

A. 164 bushels x 13 = 2132 bu.

	100	60	4	164
x 3	300	180	12	492
x 10	1000	600	40	1640
x 13	1300	780	52	2132

			1	6	4
			x	1	3
			4	9	2
		1	6	4	0
		2	1	3	2

B. $247 x 25 = $ 6175

	200	40	7	247
x 5	1000	200	35	1235
x 20	4000	800	140	4940
x 25	5000	1000	175	6175

			2	4	7
			x	2	5
		1	2	3	5
		4	9	4	0
		6	1	7	5

C. 508 pounds x 124 = 62992 lbs.

	500	0	8	508
x 4	2000	0	32	2032
x 20	10000	0	160	10160
x 100	50000	0	800	50800
x 124	62000	0	992	62992

			5	0	8
		x	1	2	4
		2	0	3	2
	1	0	1	6	0
	5	0	8	0	0
	6	2	9	9	2

D. 167 feet x 218 = 36406 feet

	100	60	7	167
x 8	800	480	56	1336
x 10	1000	600	70	1670
x 200	20000	12000	1400	33400
x 218	21800	13080	1526	36406

			1	6	7
		x	2	1	8
		1	3	3	6
		1	6	7	0
	3	3	4	0	0
	3	6	4	0	6

Tasks: *In exercises A-D pupils carry out the computations necessary to complete each table. They also record in the squares to the right of each table the partial products and their sums.*

Purpose: To apply the distributive principle to develop a multiplication algorithm for large whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: As we apply an algorithm of multiplication to larger and larger numbers we are confronted with situations that are extremely hard to visualize. We depend more and more on our conviction that the methods we found reliable for smaller numbers will be just as reliable for larger numbers.

We can help develop this confidence by selecting an example or two and starting with a sketch.

167 feet × 218 = ___ feet

We can imagine taking wire off a 167-foot roll and cutting it into three pieces—one 100 feet long, one 60 feet long, and one 7 feet long. If we did this with each of 218 rolls, we would group the rolls (and the pieces obtained by cutting each roll) with an eye toward easy computations—groups of 200, 10, and 8. We obviously want to avoid one-by-one counting, so the sketch may as well be an abbreviation:

	100	60	7	
200	(20,000)	(12,000)	(1,400)	(33,400)
10	(1,000)	(600)	(70)	(1,670)
8	(800)	(480)	(56)	(1,336)
	(21,800)	(13,080)	(1,526)	36,406

Take the time to draw an alternative sketch that leads to less efficient bookkeeping—but to the same final result:

167 feet × 218 = ___ feet

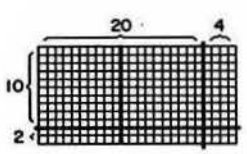
	50	50	50	17
70	(3,500)	(3,500)	(3,500)	(1,190)
70	(3,500)	(3,500)	(3,500)	(1,190)
70	(3,500)	(3,500)	(3,500)	(1,190)
8	(400)	(400)	(400)	(136)

The broken lines in the sketch help us see how multiplication can be used to simplify the addition that must still be done:

$$\begin{aligned} 9 \times 3500 &= 31{,}500 \\ 3 \times 1190 &= 3{,}570 \\ 3 \times 400 &= 1{,}200 \\ 1 \times 136 &= 136 \\ &\overline{\quad\quad 36{,}406} \end{aligned}$$

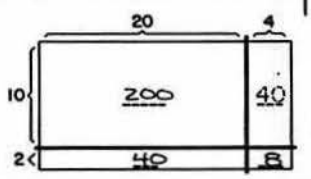
This method has its own crude kind of efficiency. More important, however, is the fact that the results are the same. This reinforces our belief in the reliability of the main strategy we are developing.

PS-42

① 

4 × 2 = 8 20 × 2 = 40 20 × 10 = 200
(20+4) × 2 = 48 20 × 2 = 40
4 × 10 = 40 20 × (10+2) = 240
(20+4) × 10 = 240 4 × (10+2) = 48
(20+4) × (10+2) = 288

② If we omit the blocks, we can use the space to keep some of the records. Here is the same sketch. You complete the records.



We can indicate combinations of the sketch this way:
(20 × 10) + (4 × 2) = 208
(4 × 10) + (20 × 2) = 80
(20+4) × 10 + (20+4) × 2 = 288
(20+4) × (10+2) = 288

③ Show the number of blocks there would be in each space.

	4	3	7	5	
3	12	9	21	15	(3+7) × 3 = 30 (7+5) × 3 = 36
6	24	18	42	30	4 × (6+4) = 40 5 × (6+4) = 50
4	16	12	28	20	(4+3) × 6 = 42 7 × (6+4) = 70
					(3+7+5) × 4 = 60 (4+3+7) × 6 = 84

(7+5) × (6+4) = 120 (3+7) × (6+4) = 100 (4+3+7+5) × 3 = 57
(4+3+7+5) × (6+4) = 190 (4+7+3+5) × (3+6+4) = 247

④ One week Helen put 36 U.S. stamps on each of 5 pages in her album. The same week she put 64 foreign stamps on each of 5 pages. That is a total of 500 stamps.
(36+64) × 5 = 100 × 5 = 500

She bought 8 stamps at 3¢, 8 stamps at 5¢, 8 stamps at 8¢, and 8 stamps at 9¢. That is a total of $2.00. 8(3+5+8+9) = 8×25 = 200
What would 1 stamp at each of the prices listed cost? 25¢.
What would 4 at each price cost? $1.00

"Mother liked to make puzzles out of practice examples. Her rules were simple: Complete each example by writing the correct digit in each empty box. Write nothing outside the boxes."

1.
		5
x		3
		5

2.
		3
x		6
1		8

3.
		9
x		3
2		7

4.
		4
x		6
2		4

5.
		7
x		7
4		9

6.
	2	5
x		4
1	0	0

7.
	1	2
x		9
1	0	8

8.
		5
x		5
2		5

9.
		7
x		8
5		6

10.
		9
x		9
8		1

11.
1		0
x		7
7		0

12.
		9
x		7
6		3

13.
	3	0
x		5
1	5	0

14.
1	8	1
x		5
9	0	5

15.
		8
x		4
3		2

16.
		8
x		8
6		4

17.
		6
x		7
4		2

18.
1		3
x		1
1		3

19.
		9
x		8
7		2

20.
1		7
x		7
1	1	9

21.
		2	3
x		5
1	1	5

22.
	5	0
x		4
2	0	0

23.
	5	3
x		4
2	1	2

24.
4	6	1
x		2
9	2	2

25.
	8	5
x		2
1	7	0

26.
	9	0
x		6
5	4	0

27.
1	2	1
x		2
2	4	2

28.
	7	0
x		7
4	9	0

29.
	4	3
x		9
3	8	7

30.
	9	9
x		5
4	9	5

31.
	8	0
x		8
6	4	0

32.
1	1	8
x		5
5	9	0

33.
1	6	9
x		3
5	0	7

34.
	1	3
x	1	2
	2	6
1	3	
1	5	6

35.
	2	6
x	1	2
	5	2
2	6	
3	1	2

36.
	2	5
x	1	4
1	0	0
2	5	
3	5	0

37.
	1	5
x	1	5
	7	5
1	5	
2	2	5

38.
	1	1
x	1	1
	1	1
1	1	
1	2	1

39.
	2	3
x	1	3
	6	9
2	3	
2	9	9

40.
	1	2	4
x	1	1	
	1	2	4
1	2	4	
1	3	6	4

41.
	2	0	7
x		2	7
1	4	4	9
4	1	4	
5	5	8	9

42.
		8	5
x		1	3
	2	5	5
	8	5	
1	1	0	5

43.
		8	0
x		7	5
	4	0	0
5	6	0	
6	0	0	0

44.
		4	3
x		4	3
	1	2	9
1	7	2	
1	8	4	9

Tasks: *Pupils carry out the necessary multiplication and division computations in order to complete each of the examples 1-44.*

Purpose: To perfect computational skills with whole numbers while exploring number puzzle forms.

Unifying Ideas: Addition and subtraction; multiplication and division.

The Lesson: "Gremlins" are allies of arithmetic teachers. After school, they come in and erase part of the arithmetic work done on the chalkboard during the day. They do a most selective job of erasing. In the morning, when curiosity over the gremlins' work has reached a healthy pitch, the teacher asks if the pupils can reconstruct what the gremlins erased.

$3+2=$ $\quad\quad$ $2+\ =6$

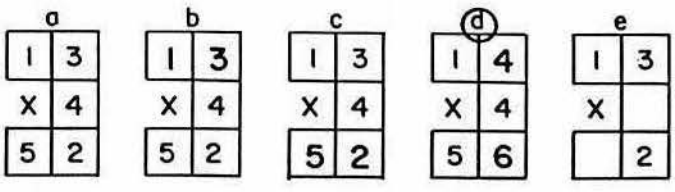
$+4=9$ $\quad\quad$ $10-\ =7$, etc.

Pupil page 43 presents a similar activity. The empty squares are places where digits were originally written. The problem is to fill in each empty square with the correct digit.

Let us consider the strategy for finding the missing digits in exercise 5. We know that one factor is 7, that the product has two digits, and that the last digit of the product is 9. Thus we are seeking a two-digit multiple of 7 with last digit 9. Simple trial and error establishes that the missing factor is 7 and the product is 49. How many products must be tested to establish that this solution is unique? (Thirteen; 2×7 through 14×7 give two-digit products.)

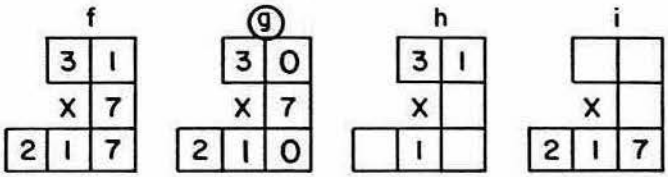
Each example on this pupil page has a unique set of correct entries for the missing digits.

An interesting variation of this work is to consider how digits in a given example can be erased so that they can be replaced in only one way. Let's inspect several ways of erasing two of the five digits in the example $13\times4=52$. Does each have a unique solution?



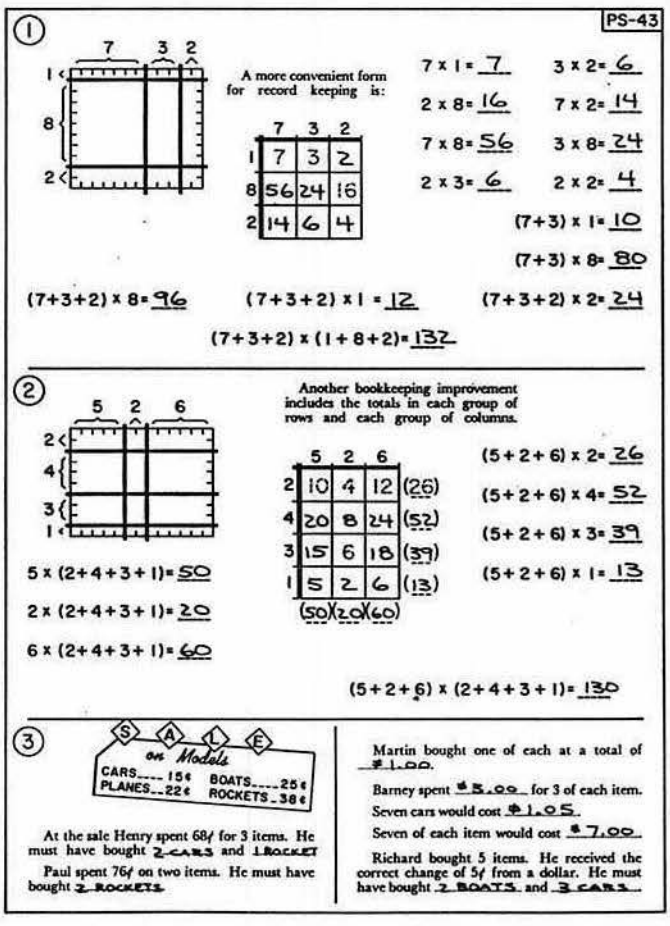
Examples *b*, *c*, and *e* have unique solutions; *d* can be solved in two ways: $13\times4=52$ and $14\times4=56$.

Try another example, leaving out three digits:



Only *g* is not unique. Multiplying any whole number 30 through 39 by 7 results in a three-digit product whose first digit is 2.

Such experimentation involves lots of practice in the basic multiplication combinations. Each example becomes a puzzle. Some pupils will try to leave out as many digits as possible. Others will make up their own "missing-digits" examples to challenge their classmates.


PS-43
A more convenient form for record keeping is:
7 x 1 = 7
3 x 2 = 6
2 x 8 = 16
7 x 2 = 14
7 x 8 = 56
3 x 8 = 24
2 x 3 = 6
2 x 2 = 4
(7+3) x 1 = 10
(7+3) x 8 = 80
(7+3+2) x 8 = 96
(7+3+2) x 1 = 12
(7+3+2) x 2 = 24
(7+3+2) x (1+8+2) = 132
Another bookkeeping improvement includes the totals in each group of rows and each group of columns.
(5+2+6) x 2 = 26
(5+2+6) x 4 = 52
(5+2+6) x 3 = 39
(5+2+6) x 1 = 13
5 x (2+4+3+1) = 50
2 x (2+4+3+1) = 20
6 x (2+4+3+1) = 60
(5+2+6) x (2+4+3+1) = 130
Models
CARS 15¢ BOATS 25¢
PLANES 22¢ ROCKETS 38¢
At the sale Henry spent 68¢ for 3 items. He must have bought 2 CARS and 1 ROCKET
Paul spent 76¢ on two items. He must have bought 2 ROCKETS
Martin bought one of each at a total of $1.00.
Barney spent $3.00 for 3 of each item.
Seven cars would cost $1.05.
Seven of each item would cost $7.00.
Richard bought 5 items. He received the correct change of 5¢ from a dollar. He must have bought 2 BOATS and 3 CARS.

"Of course, we sometimes gave Mother long lists that included the answer to the problem she gave us. Here's an example."

12 books x 15 = 180 books

(A)

Numbers of books

	2	3	7	10	12	20	22	23	27	30	37
x 5	10	15	35	50	60	100	110	115	135	150	185

Numbers of books

	2	3	7	10	12	20	22	23	27	30	37
x 10	20	30	70	100	120	200	220	230	270	300	370

Numbers of books

	2	3	7	10	12	20	22	23	27	30	37
x 15	30	45	105	150	180	300	330	345	405	450	555

"We learned to save a lot of space in making such lists of answers. I'll give you two examples. Mother might say:

16 tiles x 13 = 208 tiles

(B)

Numbers of tiles

	2	4	6	10	16	20	30	5	23	25	48
x 3	6	12	18	30	48	60	90	15	69	75	144
x 10	20	40	60	100	160	200	300	50	230	250	480
x 13	26	52	78	130	208	260	390	65	299	325	624

17 seats x 14 = 238 seats

(C)

Numbers of seats

	3	4	7	10	20	30	17	13	24	37	50
x 5	15	20	35	50	100	150	85	65	120	185	250
x 9	27	36	63	90	180	270	153	117	216	333	450
x 14	42	56	98	140	280	420	238	182	336	518	700
x 4	12	16	28	40	80	120	68	52	96	148	200
x 10	30	40	70	100	200	300	170	130	240	370	500

44-45 More on two- and three-digit multipliers

Tasks: *In exercises A-C pupils use the arrows as clues to develop shortcuts for computing the missing products.*

Purpose: To perfect computational skills while continuing to develop an algorithm for the multiplication of whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On pupil page 44 we step back to consider easy examples of multiplication so that pupils' attention can be directed once again to the distributive principle. This principle is the basis for the multiplication algorithm we are developing. The distributive principle of multiplication over addition states that, for any numbers a, b, and c,

$$(a+b)\times c=(a\times c)+(b\times c)$$

Each example on this page provides for several applications of this principle. In the top line of example A, we skip around a bit:

	2	10	12		3	20	23		30	7	37
x 5	10	50			15	100			150	35	

$$12\times5=(2+10)\times5=10+50=60$$
$$23\times5=(3+20)\times5=15+100=115$$
$$37\times5=(30+7)\times5=150+35=185$$

We also know that, for any a, b, and c,

$$a\times(b+c)=(a\times b)+(a\times c)$$

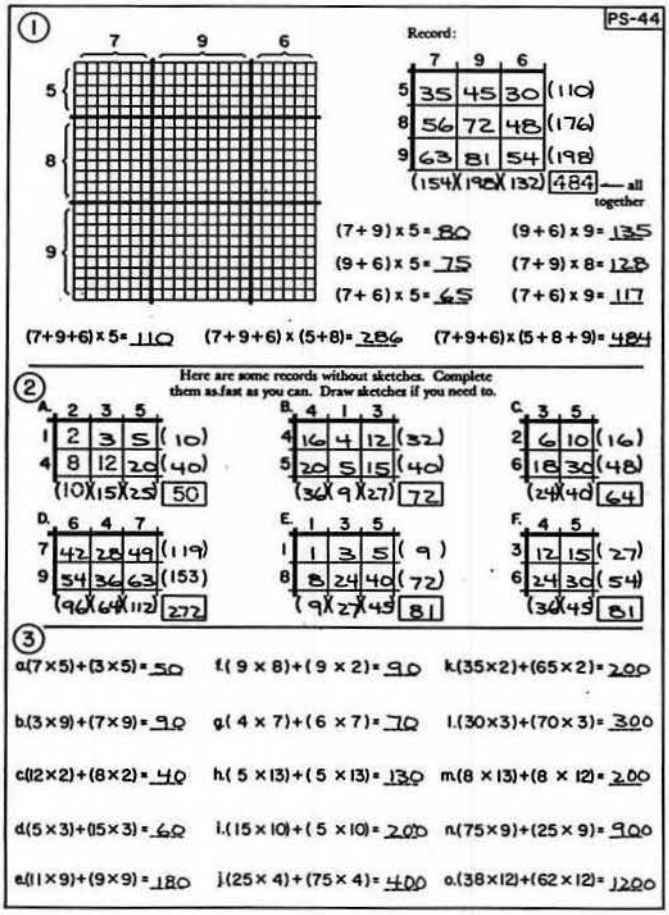
Example B is designed to point this out:

	2		6		10		16	
x 3	6		18		30		48	
x 10	20		60		100		160	
x 13								

$$2\times13=2\times(3+10)=6+20=26$$
$$6\times13=6\times(3+10)=18+60=78$$
$$10\times13=10\times(3+10)=30+100=130$$
$$16\times13=16\times(3+10)=48+160=208$$

Practice sheet 44 gives a pictorial representation of repeated use of the distributive principle to find that $22\times22=484$.

On pupil page 44 we emphasize the use of 10 as the basis for partitioning. On practice sheet 44 we point out that any partitioning leads to correct results. The question is not one of right or wrong but of which leads to the simplest computations and the easiest bookkeeping. By this standard, the use of 10 or a multiple of 10 as one addend with a second addend less than 10 is best.



A. $75 x 12 = $ 900

	10	20	30	40	70	5	75	150
x 2	20	40	60	80	140	10	150	300
x 10	100	200	300	400	700	50	750	1500
x 12	120	240	360	480	840	60	900	1800
x 4	40	80	120	160	280	20	300	600
x 8	80	160	240	320	560	40	600	1200

B. $175 x 27 = $ 4,725

	25	50	100	75	150	175
x 7	175	350	700	525	1050	1225
x 20	500	1000	2000	1500	3000	3500
x 27	675	1350	2700	2025	4050	4725

C. 87 sq. ft. x 96 = 8,352 sq. ft.

	100	10	40	80	7	87
x 6	600	60	240	480	42	522
x 90	9000	900	3600	7200	630	7830
x 96	9600	960	3840	7680	672	8352

D. $105 x 132 = $ 13,860

	5	100	105	210	315
x 2	10	200	210	420	630
x 30	150	3000	3150	6300	9450
x 100	500	10000	10500	21000	31500
x 132	660	13200	13860	27720	41580

Tasks: *Pupils use the shortcuts developed on pupil page 44 to compute the missing products.*

Purpose: To perfect computational skills while continuing to develop an algorithm for multiplication of whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Pupil page 45 is a checkup on the extension of a multiplication algorithm. The examples provide practice over a wide range of computational difficulties. After a brief discussion of the four examples you may want each pupil to show how far he can go on his own.

When considering the need for reteaching, remember that in the pupil pages and in the practice sheets, we shall be returning to the algorithm again and again—on all levels of difficulty.

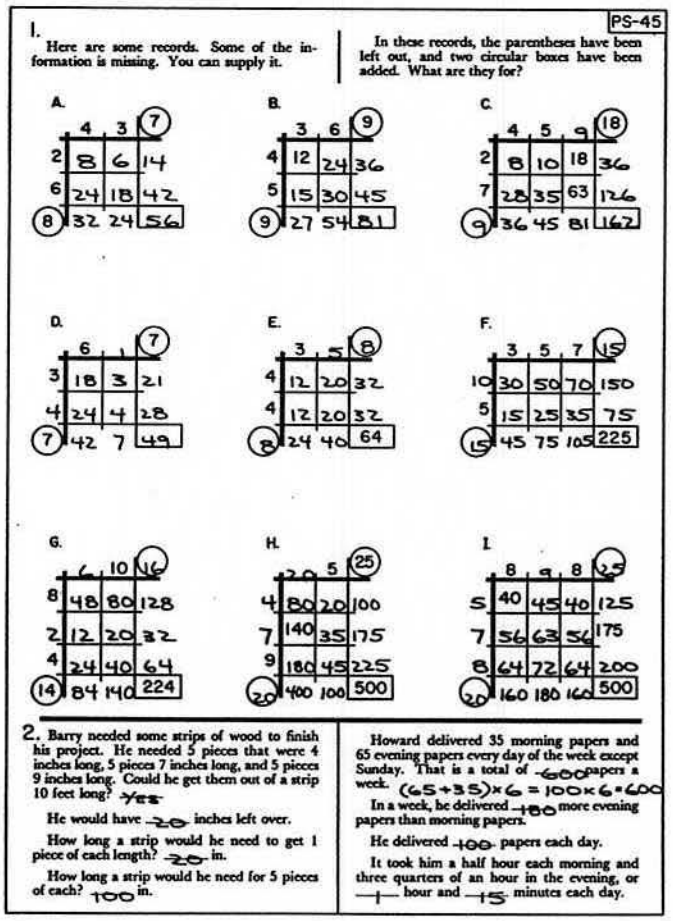
As pupils do multiplication problems involving larger numbers, the columns of numbers to be added get longer. You might remind them that it becomes increasingly important to be neat in recording their results.

Another aid in working out computations is plenty of scratch paper. Some pupils are not yet very adept at mental calculations. They ought to be encouraged to check their results. While we wish to help all children do more in their heads, we must be aware that not all children will be equally competent in performing mental computations. You should not be too dismayed if children use long forms such as:

315		175
×2		×7
10		35
20	or	490
600		700
630		1225

While these children will work a bit more slowly than others, their scratch-paper work clearly demonstrates that they understand what they are doing.

On practice sheet 45 we again demonstrate that there is no one way of regrouping to replace hard problems by a series of easy ones. We must remember that the possible regroupings are many and that the usual multiplication algorithm selected is the one that generally leads to simpler bookkeeping along the way.


PS-45
1. Here are some records. Some of the information is missing. You can supply it.
In these records, the parentheses have been left out, and two circular boxes have been added. What are they for?
A.
B.
C.
D.
E.
F.
G.
H.
I.
2. Barry needed some strips of wood to finish his project. He needed 5 pieces that were 4 inches long, 5 pieces 7 inches long, and 5 pieces 9 inches long. Could he get them out of a strip 10 feet long? Yes
He would have 20 inches left over.
How long a strip would he need to get 1 piece of each length? 20 in.
How long a strip would he need for 5 pieces of each? 100 in.
Howard delivered 35 morning papers and 65 evening papers every day of the week except Sunday. That is a total of 600 papers a week. (65+35)×6 = 100×6=600
In a week, he delivered 180 more evening papers than morning papers.
He delivered 100 papers each day.
It took him a half hour each morning and three quarters of an hour in the evening, or 1 hour and 15 minutes each day.

Mrs. Brown Raises a Question.

"I'm very happy to learn about your Arithmetic of Things and Numbers," Mrs. Brown told Ben and Ruth. "In fact, the whole class has been interested.

"As you have been explaining, I noted something I'd like to discuss.

"You say:

3 apples + 4 apples = 7 apples

3 hours + 4 hours = 7 hours

3 trucks + 4 trucks = 7 trucks

"Why can't we simply say:

$$3 + 4 = 7$$

meaning, 3 things + 4 things equals 7 things?"

Ben thought a minute. He looked at Ruth, then asked:

"But what about this?"

3 dimes + 4 nickels = _____

Jennie said, "Well, we could say that 3 dimes plus 4 nickels equals 7 coins."

"That's right," Ben agreed, "but we could also say the result was any of the following:

50 pennies

or 10 nickels

or 5 dimes

or 2 quarters

or 1 half-dollar

"It seems to me," Wanda said, "that most of the time there isn't any question what things we are talking about. So when we write:

$$3 + 4 = 7$$

we assume we are talking about the same kind of things."

Al joined the discussion. "I agree with Wanda. If there might be any question, then you ought to say exactly what it is you are talking about."

"I think they are right," Ruth admitted to Ben. "Mother always said that what we say in arithmetic ought to make sense. When we mean something special, we ought to make our meaning clear. If I write:

$$3 + 4 = 1$$

people will not understand. They will say I am wrong unless I explain that I mean:

3 days + 4 days = 1 week.

Now everyone would agree with my result."

"Perhaps we could agree," Mrs. Brown suggested, "that we will always indicate a special situation where it wouldn't otherwise be clear. Our motto can be: When in doubt, explain!"

Ben still was not happy. "How about multiplication? In addition:

Things + Things = Things.

46-47 Transition to arithmetic of numbers (when "things" are not specified)

Tasks: Pupils read and discuss with the teacher the contents of pupil page 46.

Purpose: To continue exploring the story of the *Arithmetic of Things and Numbers.*

Unifying Ideas: Number and counting; addition and subtraction.

The Lesson: As we have worked with the *Arithmetic of Things and Numbers*, the emphasis on the Things involved has gradually decreased—in preparation for the discussion on pupil pages 46 and 47. Pupils' ideas about addition and subtraction grow out of their experiences in manipulating things. Addition is essentially a shortcut for the following kind of activity:

> Count the books in one pile; count the books in another pile; put the piles together and count the books in the new pile.

The sentence

> 7 books + 5 books = 12 books

describes the activity we have carried out. If we were to experiment with other objects, we would be led to reports such as:

> 7 pencils + 5 pencils = 12 pencils
> 7 boys + 5 girls = 12 children
> 7 oranges + 5 apples = 12 pieces of fruit

While the objects vary from example to example and we may have to think about what to name the objects in the sum, we notice one aspect of all the experiments that remains constant. We point out this common aspect when we write:

$$7 + 5 = 12$$

This is our way of recording a fact we learn from experience—that the sum of five of anything and seven of anything is twelve of something.

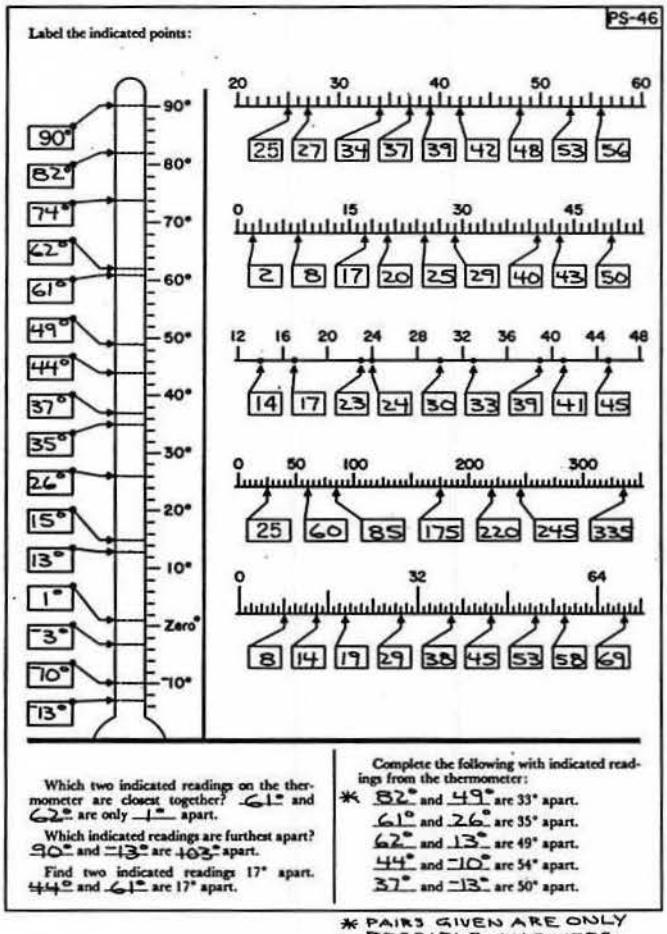
In answer to Ben's objection, we can only say that we have agreed that when we make a statement like:

$$7 + 5 = 12$$

we are describing a common aspect of otherwise different situations.

Practice sheet 46 provides a variety of scales in which not all points are labeled. You might point out that a clock is perhaps the most commonly used scale of this kind.

On the thermometer, three readings are below zero. We expect children to read "$^{-}3^{\circ}$" as 3 degrees below zero. It is not our intention at this point to discuss positive and negative numbers. We do hope however, that situations like these will lead pupils to discover that the whole numbers they have been studying are not always adequate to express ideas we encounter in real life.


PS-46
Label the indicated points:
90°
80°
70°
60°
50°
40°
30°
20°
10°
Zero°
-10°
90° 82° 74° 62° 61° 49° 44° 37° 35° 26° 15° 13° 1° -3° -10° -13°
20 30 40 50 60
25 27 34 37 39 42 48 53 56
0 15 30 45
2 8 17 20 25 29 40 43 50
12 16 20 24 28 32 36 40 44 48
14 17 23 24 30 33 39 41 45
0 50 100 200 300
25 60 85 175 220 245 335
0 32 64
8 14 19 29 38 45 53 58 69
Which two indicated readings on the thermometer are closest together? 61° and 62° are only 1° apart.
Which indicated readings are furthest apart? 90° and -13° are 103° apart.
Find two indicated readings 17° apart. 44° and 61° are 17° apart.
Complete the following with indicated readings from the thermometer:
* 82° and 49° are 33° apart.
61° and 26° are 35° apart.
62° and 13° are 49° apart.
44° and -10° are 54° apart.
37° and -13° are 50° apart.
* PAIRS GIVEN ARE ONLY POSSIBLE ANSWERS.

"But in multiplication:

Things x a Number = Things

"Here are some examples:

3 days	x 4 =	1 week and 5 days
3 inches	x 4 =	1 foot
3 bags	x 4 =	12 bags

"Don't we need to know what we are multiplying?" Ben asked.

Tom joined in. "But there is something the same about Ben's three examples. I could write them this way:

3 days	x 4 =	12 days
3 inches	x 4 =	12 inches
3 bags	x 4 =	12 bags

and the idea they all contain is what I mean when I write:

$$3 \times 4 = 12$$

"Isn't that reasonable, Ben?"

"Not yet. When you write it as you have, how do I know whether the 3 refers to things or to a number to multiply by? You might be thinking of:

3 days x 4 = 12 DAYS

or

3 x 4 days = 12 DAYS

3 days multiplied by 4, or 3 times 4 days."

Tom had his answer ready: "But the answer is 12 days or 1 week and 5 days either way."

"You are right," Ben had to admit, "but I'm still not too happy. When I see:

$$6 \times 4 = 24$$

I don't know whether you started with 6 things or with 4 things."

"But you do know," Tom argues, "that the result is 24 things in either case."

"I see Ben's point," Mary said. "I think when we want people to remember what things we are talking about, we should tell them. Otherwise we don't need to write out the whole story."

Mrs. Brown had an idea. "I'm going to write down some questions in arithmetic. I'll not tell you what things I'm thinking about, nor will I tell you when I am thinking about numbers or when I'm thinking about things. How do you think I would complete each of the following statements?"

A. $17 + 5 = 22$ B. $8 \times 3 = 24$

C. $16 \div 2 = 8$ D. $30 - 9 = 21$

E.
$$\begin{array}{r} 13 \\ \times\ 3 \\ \hline 39 \end{array}$$

F.
$$\begin{array}{r} 37 \\ +28 \\ \hline 65 \end{array}$$

G.
$$\begin{array}{r} 25 \\ \times\ 3 \\ \hline 75 \end{array}$$

H.
$$4\overline{)24}\ 6$$

"Suppose I had been thinking of marbles, or inches, or elephants, or pennies. Would you consider that your correct answers would make sense?" YES.

Tasks: Pupils read and discuss pupil page 47. In exercises A-H pupils compute the indicated sums, differences, products, and quotients.

Purpose: To continue exploring the story of the *Arithmetic of Things and Numbers.*

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

The Lesson: Counting things and recording results often leads to consideration of several groups of quantities of the same size.

3 seats to a row; 4 rows
3 groups of pictures; 5 in a group
30¢ per book; we buy 5 books

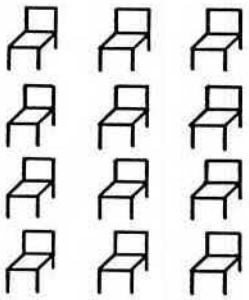
We could record our results this way:

3 seats + 3 seats + 3 seats + 3 seats = 12 seats
5 pictures + 5 pictures + 5 pictures = 15 pictures
etc.

But we prefer to do it this way:

3 seats × 4 = 12 seats
5 pictures × 3 = 15 pictures

If we arrange the objects in a rectangular array:



we can view them as 4 rows with 3 seats in a row or as 3 rows with 4 seats in a row, and write:

4 seats × 3 = 12 seats
3 pictures × 5 = 15 pictures

We notice that if seats or pictures are replaced in the arrays by other objects, our reports will all have a common aspect which we describe by writing:

$3 \times 4 = 12$ $\quad$ $5 \times 3 = 15$
$4 \times 3 = 12$ $\quad$ $3 \times 5 = 15$, etc.

Again, we must assure Ben that when we do not specify what things we are talking about we are interested only in how many, not what.

By labeling all numbers divisible by 2, 3, 5, and 7 less than 101 in the chart on practice sheet 47, we leave unlabeled all the prime numbers less than 101 (except 2, 3, 5, and 7). In fact, we could extend the chart to 120 without missing a prime. To go beyond 120 we would have to add multiples of 11 to our list. (All multiples of 11 less than 121 are already labeled as multiples of 2, 3, 5, or 7.) This would suffice until we reach 169 when 13 would have to be added, etc.

Every whole number greater than 1 that is not a prime number can be written as a product of prime numbers in only one way. For example, there is only one way to express 36 as a product of prime numbers: $36 = 2 \times 2 \times 3 \times 3$. This is a fundamental idea that pupils will encounter often.

PS-47

Which numbers are divisible by 2, 3, 5, or 7?
Mark each box under a number:
"A" . . . if divisible by 2 | "C" . . . if divisible by 5
"B" . . . if divisible by 3 | "D" . . . if divisible by 7

1	2	3	4	5	6	7	8	9	10
	A	B	A	C	AB	D	A	B	AC
11	12	13	14	15	16	17	18	19	20
X	AB	X	AD	BC	A	X	AB	X	AC
21	22	23	24	25	26	27	28	29	30
BD	A	X	AB	C	A	B	AD	X	ABC
31	32	33	34	35	36	37	38	39	40
X	A	B	A	CD	AB	X	A	B	AC
41	42	43	44	45	46	47	48	49	50
X	ABD	X	A	BC	A	X	AB	D	AC
51	52	53	54	55	56	57	58	59	60
B	A	X	AB	C	AD	B	A	X	ABC
61	62	63	64	65	66	67	68	69	70
X	A	BD	A	C	AB	X	A	B	ACD
71	72	73	74	75	76	77	78	79	80
X	AB	X	A	BC	A	D	AB	X	AC
81	82	83	84	85	86	87	88	89	90
B	A	X	ABD	C	A	B	A	X	ABC
91	92	93	94	95	96	97	98	99	100
D	A	B	A	C	AB	X	AD	B	AC

Place an "X" below all numbers larger than 1, not divisible by either 2, 3, 5, or 7. List these numbers below — following 2, 3, 5, and 7.

2 3 5 7 11 13 17 19 23 29 31 37 41 43
47 53 59 61 67 71 73 79 83 89 97 (These are called PRIME NUMBERS.)

List those for which you wrote 3 different letters.

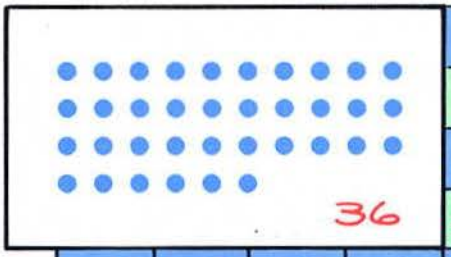
30 42 60 70 84 90

An Experiment.

Mrs. Brown prepared the material on the next 4 pages and explained to the class:

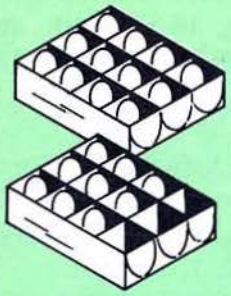
"Here is a strange mixture of numbers and things. Please fill in the blanks. If there is anything confusing, we'll talk about it."

A



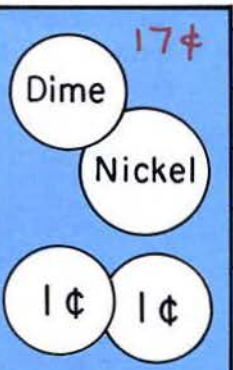
				+4	−10
				40	26
				+16	+63
				52	99
×2	÷2	+15	÷4	−29	+29
72	18	51	9	7	65

B



		+2
		23
		−17
		4
		−20
		1
×2	+19	÷3
42	40	7

C



		+8¢
		25¢
		+26¢
		43¢
		×3
		51¢
×2	×4	+95¢
34¢	68¢	112¢

D

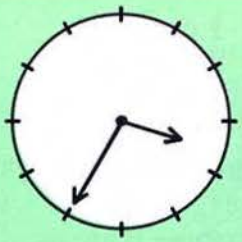
		÷2	−44
		72	100
		×1	×2
		144	288
no. of dozens		+16	−16
12		160	128
÷12	+57	−75	−97
12	201	69	47

E

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	

weeks and days tinted green	
1 wk., 5 da.	
twice as many	
3 wk., 3 da.	
1 wk., 3 da. more	half as many
3 wk., 1 da.	0 wk., 6 da.
×3	÷3
5 wk., 1 da.	0 wk., 4 da.

F



time shown	
3:35 PM	
2½ hours later	
6:05 P.M.	
33 minutes later	10 hrs. later
4:08 P.M.	1:35 A.M.
1½ hrs. earlier	5 hrs. earlier
2:05 P.M.	10:35 A.M.

G

half as many	25	thirty-three less	17
three times as many	150	twice as many	100
sixty-seven more	117	×4	200

Tasks: *In each exercise pupils determine the number represented by the picture, and then compute to determine the missing numbers.*

Purpose: To explore a variety of situations (many of which are applications) in which to practice computational skills.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Ben is still worried about dropping certain explanations in his *Arithmetic of Things and Numbers* in spite of stated agreements designed to avoid confusion. So Mrs. Brown prepared a four-page experiment—pupil pages 48 through 51. In each example, the subject under discussion is suggested by a sketch. Around the sketch are directions such as "add four," "divide by two," "subtract seven," and "multiply by three" written in shorthand:

$+4$, $\div 2$, -7, and: $\times 3$

Does any confusion result? This will be a kind of test to find out whether Ben's fears are justified.

Can we tell from the context of a situation just what is being talked about? Can our common sense help us carry out directions as they were intended?

In the first example we notice 36 dots arranged in 3 rows of 10 each and one row of 6. What could the single expression "$+4$" mean? Ben might say he doesn't know; should he add four rows, four columns, four dots, or four other sketches with 36 dots in each? Quite clearly it is intended that we add four dots so there will a total of 40. If it had been intended that we add 4 more rows or columns it would have been specified which rows or which columns, since those shown contain different numbers of dots. If the directions were intended to be interpreted as calling for 4 more sketches, what could the next direction—"-10"—mean?

Take time to talk about several of the directions. Test them for ambiguity. Then look for clues that suggest the intended meaning.

Practice sheet 48 may be introduced as the result of erasures by gremlins (see Notes on Pupil Page 43). When these gremlins erase a number, they put a frame in its place. If the same number is erased in more than one place in an example they may use frames of the same shape or of different shapes to indicate this. If the numbers are different, they must use frames of different shapes. Thus, if they leave

$$\square + \square = 36$$

we know that they must have started with

$$18 + 18 = 36$$

But if they leave

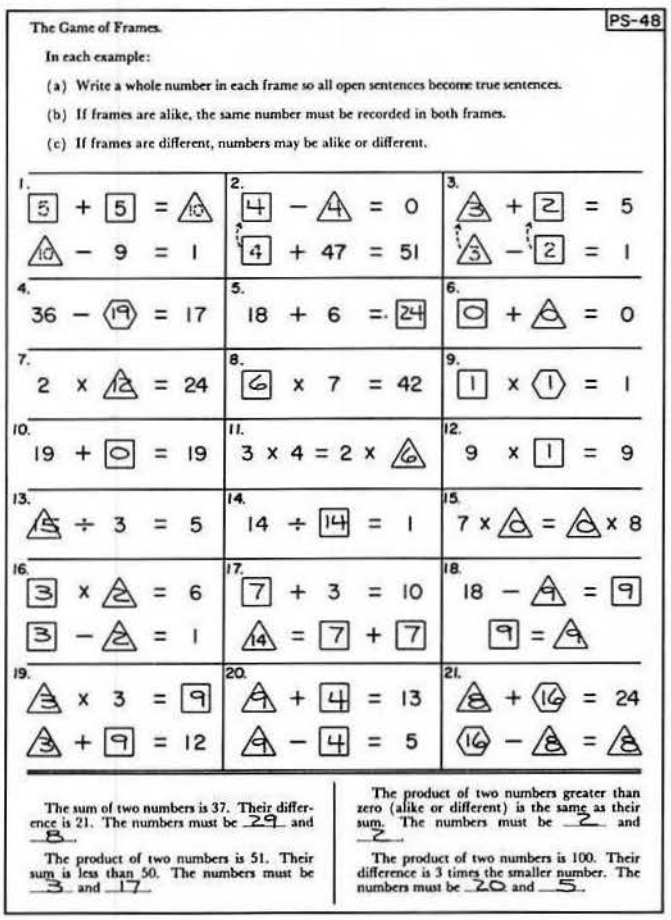
$$\square + \triangle = 36$$

they might have started with

$$9 + 27 = 36$$

or $$18 + 18 = 36$$

or any other pair of numbers whose sum is 36.



The Game of Frames. PS-48

In each example:

(a) Write a whole number in each frame so all open sentences become true sentences.

(b) If frames are alike, the same number must be recorded in both frames.

(c) If frames are different, numbers may be alike or different.

1. 5 + 5 = 10; 10 − 9 = 1
2. 4 − △ = 0; 4 + 47 = 51
3. △ + 2 = 5; 3 − 2 = 1
4. 36 − ⬠ = 17
5. 18 + 6 = 24
6. ○ + △ = 0
7. 2 × △ = 24
8. 6 × 7 = 42
9. 1 × ⬡ = 1
10. 19 + ○ = 19
11. 3 × 4 = 2 × △
12. 9 × 1 = 9
13. △ ÷ 3 = 5
14. 14 ÷ 14 = 1
15. 7 × ○ = ○ × 8
16. 3 × △ = 6; 3 − △ = 1
17. 7 + 3 = 10; 14 = 7 + 7
18. 18 − △ = 9; 9 = △
19. △ × 3 = 9; △ + 9 = 12
20. △ + 4 = 13; △ − 4 = 5
21. △ + 16 = 24; 16 − △ = △

The sum of two numbers is 37. Their difference is 21. The numbers must be 29 and 8.

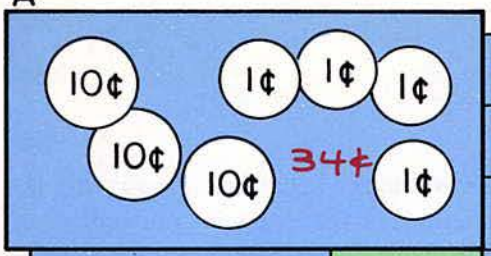
The product of two numbers is 51. Their sum is less than 50. The numbers must be 3 and 17.

The product of two numbers greater than zero (alike or different) is the same as their sum. The numbers must be 2 and 2.

The product of two numbers is 100. Their difference is 3 times the smaller number. The numbers must be 20 and 5.

A

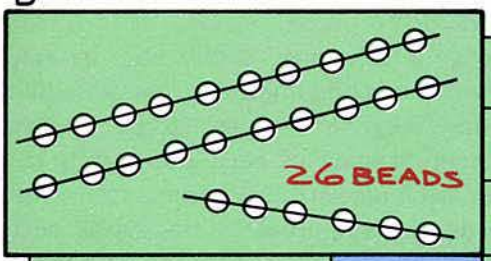
What are the coins worth?



dimes shown	30 ¢	three times as many dimes	90 ¢
pennies shown	4 ¢	three times as many pennies	12 ¢
amount shown	34 ¢	three times amount shown	$1.02
pennies shown × 7	28 ¢	pennies shown × 10	40 ¢	pennies × 17	68 ¢
dimes shown × 7	$2.10	dimes × 10	$3.00	dimes × 17	$ 5.10
all coins × 7	$ 2.38	all coins × 10	$3.40	all coins × 17	$ 5.78

B

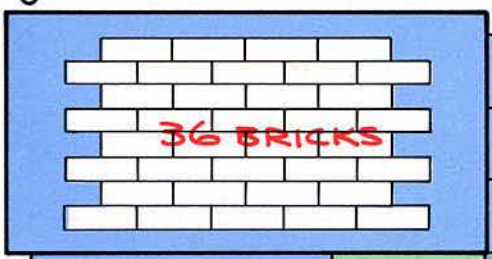
How many beads?



number shown	26	twice as many	52
half as many	13	four times as many	104
nine more	35	nine less	17
3 times as many	78	5 times as many	130	8 times as many	208
17 more	43	200 more	226	217 more	243
1 less	25	40 more	66	39 more	65

C

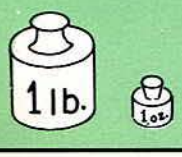
How many bricks?



bricks shown	36	9 more	45
7 more	43	79 more	115
57 more	93	279 more	315
twice as many	72	× 4	144	÷ 2	18
ten times as many	360	× 5	180	÷ 4	9
12 times as many	432	× 9	324	÷ 6	6

D

There are 16 ounces in one pound.



3 pounds	48 oz.	$1\frac{1}{2}$ pounds	24 oz.
10 pounds	160 oz.	$\frac{1}{4}$ pound	4 oz.
13 pounds	208 oz.	$1\frac{3}{4}$ pounds	28 oz.

Tasks: In each exercise pupils determine the number represented by the picture, and then compute (using the distributive principle when possible) to determine the missing numbers.

Purpose: To perfect whole-number computational skills while exploring a variety of applications.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Pupil page 49 offers several opportunities to use the distributive principle to avoid tiresome computations. Directly under the sketch of 3 dimes and 4 pennies, three questions are posed:

$4¢ \times 7 = \underline{28¢}$
$30¢ \times 7 = \underline{\$2.10}$
$34¢ \times 7 = \underline{\$2.38}$

We hope many pupils will add 28¢ and \$2.10 rather than multiply 34¢ by 7. The computations called for in the bottom line in this example are:

$34¢ \times 7 = \underline{\$2.38}$
$34¢ \times 10 = \underline{\$3.40}$
$34¢ \times 17 = \underline{\$5.78}$

Again, we hope many children will add \$2.38 and \$3.40 rather than multiply 34¢ by 17.

Five of the questions of exercise B are listed in a different order and form in lines *b* through *f* below:

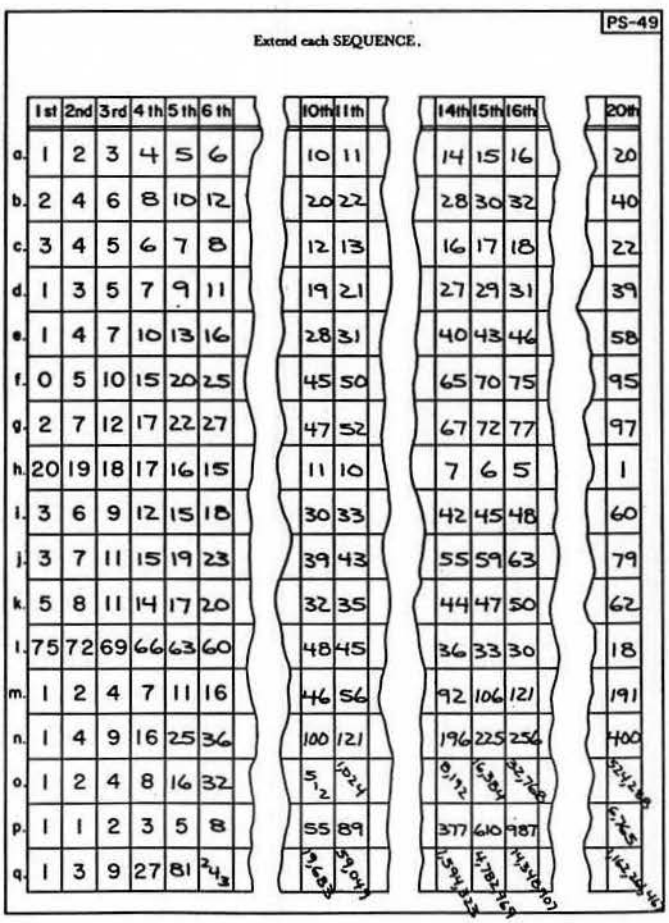
(a) $26 \times 1 = \underline{26}$
(b) $26 \times 2 = \underline{52}$
(c) $26 \times 3 = \underline{78}$
(d) $26 \times 4 = \underline{104}$
(e) $26 \times 5 = \underline{130}$
(f) $26 \times 8 = \underline{208}$

Line *b* calls for twice the result in line *a*. Line *c* calls for the sum of the results in lines *a* and *b*. Line *d* calls for the sum of the results in lines *a* and *c* or for twice the result in line *b*. Line *e* calls for the sum of results in lines *b* and *c*. Line *f* calls for twice the result in line *d*.

Looking for such relationships is a useful strategy that builds on what we have learned about relationships in arithmetic.

Extending sequences such as those on practice sheet 49 provides other opportunities to use relationships. The sequence 1, 2, 3, . . . can be extended simply by noting the number of the term; clearly 14 will be the 14th term, 16 the 16th, etc. The sequence 2, 4, 6, . . . can be extended by noting that each term is twice the number of the term. Each term of 3, 4, 5, . . . is two more than the number of the term—16 is the 14th term, 18 the 16th, etc. Each term of 1, 4, 7, . . . is 2 less than 3 times the number of the term; so the 20th term is $(20 \times 3) - 2 = 58$.

However, if the relationship between each term and its number cannot be found, writing out the first 20 terms is a sure way of finding the needed terms. (In fact, the sequence 1, 1, 2, 3, 5, . . . in which each term is the sum of the two preceding terms, is best done in this way.)

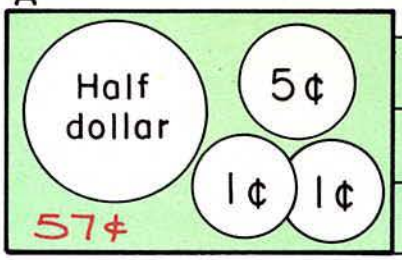


PS-49

Extend each SEQUENCE.

	1st	2nd	3rd	4th	5th	6th	10th	11th	14th	15th	16th	20th
a.	1	2	3	4	5	6	10	11	14	15	16	20
b.	2	4	6	8	10	12	20	22	28	30	32	40
c.	3	4	5	6	7	8	12	13	16	17	18	22
d.	1	3	5	7	9	11	19	21	27	29	31	39
e.	1	4	7	10	13	16	28	31	40	43	46	58
f.	0	5	10	15	20	25	45	50	65	70	75	95
g.	2	7	12	17	22	27	47	52	67	72	77	97
h.	20	19	18	17	16	15	11	10	7	6	5	1
i.	3	6	9	12	15	18	30	33	42	45	48	60
j.	3	7	11	15	19	23	39	43	55	59	63	79
k.	5	8	11	14	17	20	32	35	44	47	50	62
l.	75	72	69	66	63	60	48	45	36	33	30	18
m.	1	2	4	7	11	16	46	56	92	106	121	191
n.	1	4	9	16	25	36	100	121	196	225	256	400
o.	1	2	4	8	16	32	512	1024	8192	16384	32768	524288
p.	1	1	2	3	5	8	55	89	377	610	987	6765
q.	1	3	9	27	81	243	19683	59049	1594323	4782969	14348907	3486784401

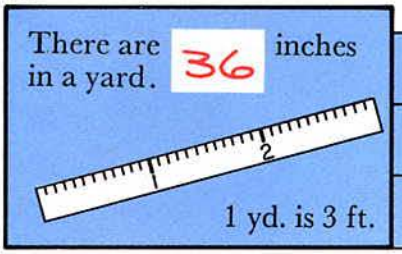
A



What are the coins worth?

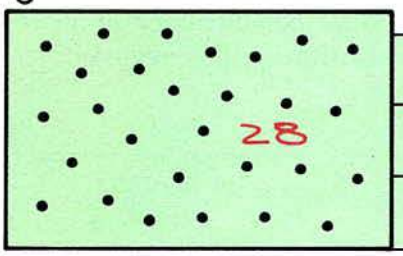
35¢ more	92 ¢	3 times as much	$ 1.71
$8.35 more	$ 8.92	10 times as much	$ 5.70
$8.70 more	$ 9.27	13 times as much	$ 7.41

B



3 yards are	9 ft.	2 yards are	72 in.
½ yard is	18 in.	10 yards are	360 in.
7 yards are	21 ft.	12 yards are	432 in.

C



Number of dots

twice as many	56	10 times as many	280
3 times as many	84	20 times as many	560
5 times as many	140	30 times as many	840

D

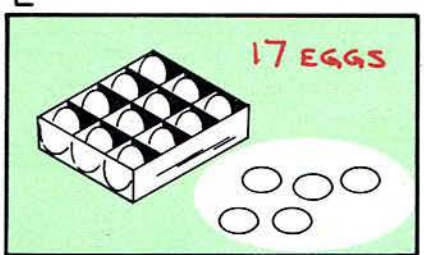
A row of B's 10
B B B B B B B B B B

A row of M's 8
M M M M M M M M

How many letters

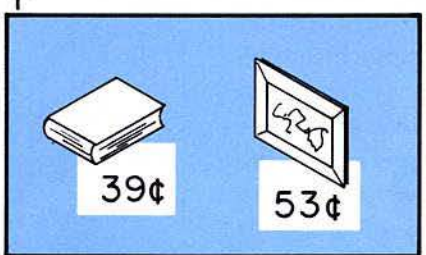
4 rows of B's	40	10 rows of each	180
4 rows of M's	32	14 rows of each	252
4 rows of each	72	7 rows of each	126

E



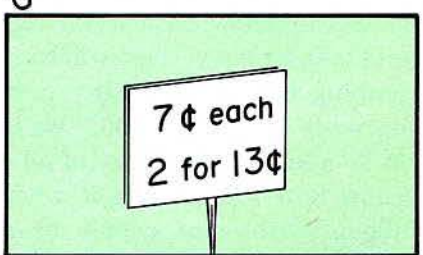
twice as many eggs
2 doz., 10 eggs
5 times as many
7 doz., 1 eggs
7 times as many
9 doz., 11 eggs

F



a book and a picture
92 ¢
3 books and a picture
$ 1.70
2 books and 3 pictures
$ 2.37

G



price of 5
33 ¢
price of 7
46 ¢
price of 12
78 ¢

Notes on Pupil Page 50

Tasks: In each exercise pupils determine the number represented by the picture, and compute (using the distributive principle where possible) to determine the missing numbers.

Purpose: To perfect whole-number computational skills while exploring a variety of applications.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: There are many opportunities for taking shortcuts on pupil page 50. In example A, if $57¢+35¢=92¢$ then $57¢+\$8.35$ is \$8 more, or \$8.92, since, using the rearrangement principle:

$$\begin{aligned} 57¢+\$8.35 &= 57¢+(\$8+35¢) \\ &= \$8+(57¢+35¢) \end{aligned}$$

Further,

$$\begin{aligned} 57¢+\$8.70 &= 57¢+(\$8.35+\$.35) \\ &= (\$8.35+57¢)+\$.35 \\ &= \$8.92+\$.35 \\ &= \$9.27 \end{aligned}$$

In example B, if

2 yards = 72 inches,
and 10 yards = 360 inches,
then 12 yards = 432 inches,
because 72 + 360 = 432.

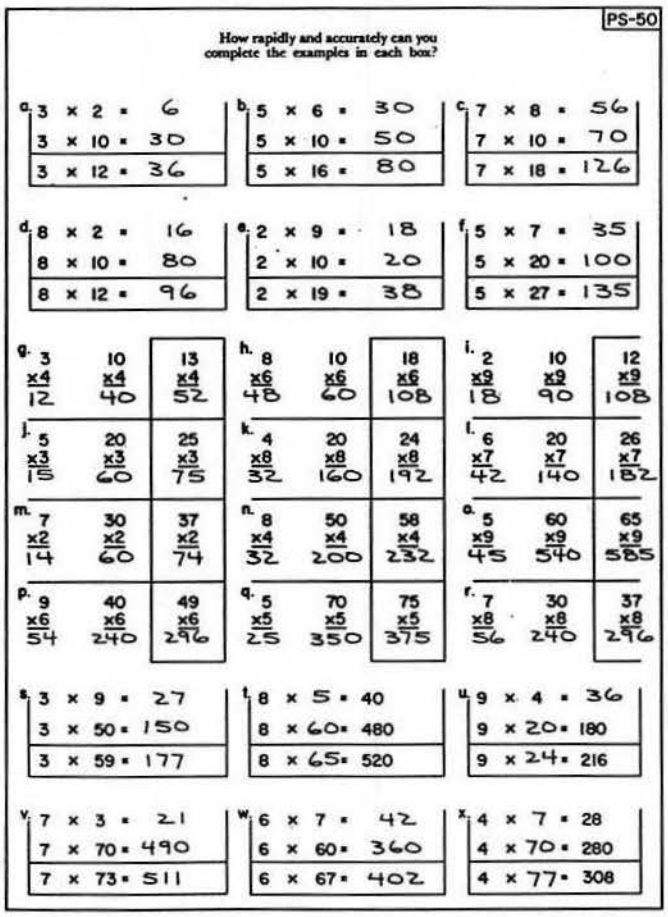
In example C, some pupils will work this way:

2 times as many 56 (28 + 28 = 56)
3 times as many 84 (56 + 28 = 84)
5 times as many 140 (2 + 3) x 28 = 140
10 times as many 280 (twice 140)
20 times as many 560 (twice 280)
30 times as many 840 (10 + 20) x 28 = 840

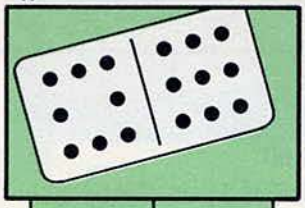
Exercise G requires special attention to find a shortcut—and to avoid the trap. After finding that 5 will cost 33¢, it is easy to see that 7 will cost 13¢ more or 46¢. But one must avoid the temptation to say that 5+7 would cost 33¢+46¢ or 79¢. Because of the way the discount is computed, the cost will be only 78¢.

Practice sheet 50 is the first of four reviewing the use of the distributive principle in multiplication. The level of difficulty increases with each page, starting at the most elementary level. Notice that pupils are asked to complete the examples in the boxes. We recommend that you tell them that they can skip the other exercises unless they need them as clues. These simply represent steps in the algorithm that appears in the boxes. They should be available for those who have not mastered the final algorithm, but should not be required of those who need no such help.

This four-page development builds step by step to examples with one three-digit and one two-digit factor. You may wish to use these pages to help determine the proficiency of each pupil with the usual multiplication algorithm. We hope that most children can do the examples in the boxes with little use of the helping steps.

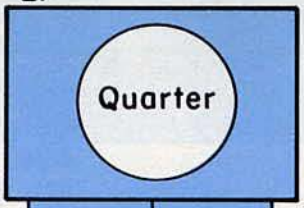


1.



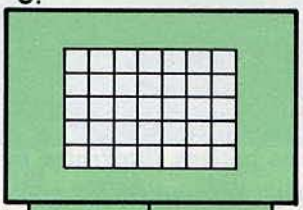
+8	−9
25	8
x2	x3
34	51
x4	x6
68	102

2.



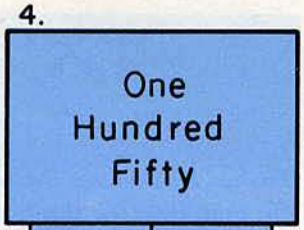
x2	x3
50¢	75¢
+18¢	−6¢
43¢	19¢
+90¢	x6
$1.15	$1.50

3.



+27	−22
62	13
+127	x2
162	70
x10	x12
350	420

4.



−50	x2
100	300
+75	−75
225	75
÷2	x3
75	450

5.

Seven more than twenty-five	+8	x2	+109
	40	64	141
÷2	−32	x3	÷4	−18
16	0	96	8	14

6.

8	x4	x10	x14	x28
	32	80	112	224
÷2	+27	−6	÷4	x9
4	35	2	2	72

7.

	10	90	47	52	85
+28	38	118	75	80	113

8.

	3	2	5	7	12
x9	27	18	45	63	108

9.

+6	x2	÷2
30	48	12
−17	3x8 (24)	÷4
7		6
+26	+76	÷8
50	100	3

10.

+7	−29	+372
50	14	415
x2	43	+20
86		63
x4	x10	x14
172	430	602

11.

+40	x2	+79	÷2
100	120	139	30
÷1	60	÷4	÷5
60		15	12
÷3	−27	÷6	÷10
20	33	10	6

"You didn't have many questions to ask. I tried to give you clues when I thought you might not understand my questions."

Ben had a comment. "That was a big mixture of things and numbers, and numbers without any particular things. But I knew all the time what you meant."

"That's fine," Mrs. Brown concluded. "We will come back to our Arithmetic of Things and Numbers when we talk about fractions."

Notes on Pupil Page 51

Tasks: In exercises 1-10 pupils determine the number represented and compute to determine the missing numbers. In exercises 7 and 8 they perform the indicated operations on the given numbers to find missing sums (or addends) and products (or factors). In exercises 9 and 10 they follow directions and operate on the number in the large box to complete the table. In exercise 11 they use the clues provided to find the number for the large box and follow directions to complete the table.

Purpose: To perfect computational skills with whole numbers.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations.

The Lesson: This page concludes the experiment prepared to find out if Ben's fears were well-founded. We hope pupils will conclude that confusion is unlikely if they use common sense in interpreting questions. Absolutely unambiguous language is rarely attainable in practice, and it would be foolish to teach children to expect it. Rather, we urge children to use common sense and to expect it of others. When questions still arise, we discuss them.

On pupil page 51 we move from sketches of things to numbers indicated by written words to numbers indicated by Hindu-Arabic numerals. When we are given the number "one hundred fifty" and the direction "−50" we are not confused, because we know that if we have 150 of anything and take 50 away, there will be 100 left. So, although nothing is said about any particular things when we write "150 − 50 = 100," we know this statement to be a kind of summary of all our experiences. We know how to interpret it whether we are talking about marbles or people or dollars or miles or minutes.

We will return to the *Arithmetic of Things and Numbers* to help us develop the idea of fractions and of addition and multiplication of fractions. We think you will find further reasons for continuing this idea as a theme in Level D.

On practice sheet 51 we suggest that pupils b urged to do only the examples in the boxes; thos who are still hestitant about the algorithm can d the two examples to the left of each box and ad the results.

$$\begin{array}{r} 3 \\ \times 6 \\ \hline 18 \end{array} \qquad \begin{array}{r} 50 \\ \times 6 \\ \hline 300 \end{array} \qquad \begin{array}{r} 53 \\ \times 6 \\ \hline 318 \end{array}$$

$$18 + 300 = 318$$

The examples in the boxes are chosen to provid practice with the most often missed combinations Any weaknesses in basic combinations or in th one-digit multiplier algorithm will be revealed.

PS-51

a. 3 × 6 = 18	50 × 6 = 300	53 × 6 = 318
b. 4 × 7 = 28	50 × 7 = 350	54 × 7 = 378
c. 3 × 9 = 27	40 × 9 = 360	43 × 9 = 387
d. 9 × 4 = 36	60 × 4 = 240	69 × 4 = 276
e. 6 × 6 = 36	40 × 6 = 240	46 × 6 = 276
f. 5 × 7 = 35	60 × 7 = 420	65 × 7 = 455
g. 4 × 5 = 20	40 × 5 = 200	44 × 5 = 220
h. 9 × 5 = 45	80 × 5 = 400	89 × 5 = 445
i. 8 × 9 = 72	70 × 9 = 630	78 × 9 = 702
j. 6 × 4 = 24	90 × 4 = 360	96 × 4 = 384
k. 9 × 8 = 72	70 × 8 = 560	79 × 8 = 632
l. 7 × 7 = 49	90 × 7 = 630	97 × 7 = 679
m. 9 × 9 = 81	90 × 9 = 810	99 × 9 = 891
n. 5 × 4 = 20	80 × 4 = 320	85 × 4 = 340
o. 6 × 8 = 48	90 × 8 = 720	96 × 8 = 768
p. 8 × 2 = 16	70 × 2 = 140	78 × 2 = 156
q. 3 × 8 = 24	80 × 8 = 640	83 × 8 = 664
r. 5 × 6 = 30	80 × 6 = 480	85 × 6 = 510
s. 8 × 8 = 64	50 × 8 = 400	58 × 8 = 464
t. 6 × 3 = 18	80 × 3 = 240	86 × 3 = 258
u. 7 × 9 = 63	50 × 9 = 450	57 × 9 = 513
v. 4 × 6 = 24	70 × 6 = 420	74 × 6 = 444
w. 3 × 5 = 15	90 × 5 = 450	93 × 5 = 465
x. 8 × 5 = 40	50 × 5 = 250	58 × 5 = 290
y. 7 × 3 = 21	90 × 3 = 270	97 × 3 = 291
z. 6 × 9 = 54	80 × 9 = 720	86 × 9 = 774
A. 8 × 3 = 24	60 × 3 = 180	68 × 3 = 204
B. 4 × 4 = 16	60 × 4 = 240	64 × 4 = 256
C. 2 × 6 = 12	90 × 6 = 540	92 × 6 = 552
D. 9 × 7 = 63	80 × 7 = 560	89 × 7 = 623

$27 \div 3 = \boxed{15} - 6$

(1) 20 (2) 15 (3) 24 (4) 9

Mark correct answer → (2) ● Mark furthest from correct answer → (3) ⊗

(1) ○ (2) ● (3) ⊗ (4) ○ 9 ← Difference between answers you marked

1 $\boxed{25} = 100 \div 4$

(1) 25 (2) 40 (3) 96 (4) 104

(1) ● (2) ○ (3) ○ (4) ⊗ 79

2 $7 \times 6 = 8 + \triangle 34$

(1) △ 67 (2) △ 34 (3) △ 42 (4) △ 5

(1) ⊗ (2) ● (3) ○ (4) ○ 33

3 $\bigcirc 8 \times 5 = 4 \times 10$

(1) ○ 20 (2) ○ 19 (3) ○ 40 (4) ○ 8

(1) ○ (2) ○ (3) ⊗ (4) ● 32

4 $32 \div 2 = \boxed{4} \times \boxed{4}$

(1) 8 (2) 16 (3) 4 (4) 30

(1) ○ (2) ○ (3) ● (4) ⊗ 26

5 $13 + \triangle 29 = 17 + 25$

(1) △ 42 (2) △ 29 (3) △ 65 (4) △ 0

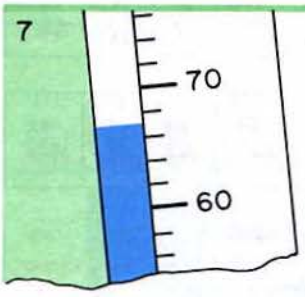
(1) ○ (2) ● (3) ⊗ (4) ○ 36

6 $\boxed{18} - 18 = \boxed{18} \times 0$

(1) 0 (2) 18 (3) 36 (4) 81

(1) ○ (2) ● (3) ○ (4) ⊗ 63

7



The temperature shown on the thermometer is

(1) 50° (3) $68\frac{1}{2}°$

(2) $63\frac{1}{2}°$ (4) 67°

(1) ⊗ (2) ○ (3) ○ (4) ● 17°

8

If the temperature dropped 19°, it would be

(1) 38° (3) 48°

(2) 0° (4) 86°

(1) ○ (2) ⊗ (3) ● (4) ○ 48°

9.

The time shown on the clock is

(1) 11:40 A.M. (3) 7:55 A.M.

(2) 11:38 A.M. (4) 8:05 A.M.

(1) ⊗ (2) ○ (3) ● (4) ○ 3 hrs., 45 min.

10

How long is it until the clock will next read 12:00?

(1) 5 hrs., 50 min.

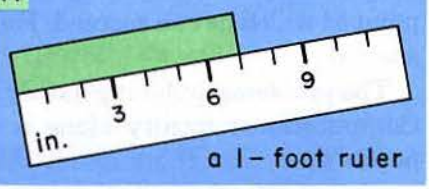
(2) 4 hrs., 5 min.

(3) 8 hrs., 55 min.

(4) 1 hr., 20 min.

(1) ○ (2) ● (3) ⊗ (4) ○ 4 hrs., 50 min.

11



a 1-foot ruler

The length of a block 4 times as long would be

(1) 1 yd., 0 ft., 4 in.

(2) 0 yd., 2 ft., 4 in.

(3) 0 yd., 0 ft., 7 in.

(4) 0 yd., 1 ft., 9 in.

(1) ○ (2) ● (3) ⊗ (4) ○ 1 ft., 9 in.

12

How many inches would have to be added to the green block in the sketch (in exercise 11) to make it 1 yard long?

(1) 29 (3) 23

(2) 36 (4) 31

(1) ● (2) ⊗ (3) ○ (4) ○ 7

52-53 Various forms of typical multiple-choice questions, with an additional twist

Tasks: *Pupils use the response pattern illustrated by the sample at the top of the page to complete the exercises.*

Purpose: To test achievement in computational skills with whole numbers.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: This pupil page and pupil page 53 serve as tests or checkups using the multiple-choice form pupils will meet in most standardized achievement tests. If the class is not already familiar with them, the boxes in such open sentences as $\square = 100 \div 4$ can be introduced as the result of erasures by gremlins (see Notes on Pupil Page 48). Remember the rules: When frames of the same shape occur in a sentence, they must be filled with the same number. Frames of different shape in the same sentence may be filled with the same or different numbers. Unless pupils understand the first of these rules, they cannot be expected to understand questions 4 and 6.

If the class has difficulty with questions 7 through 12, they should have further experiences with scales (see practice sheet 46) and with problems involving readings of scales. You might want to prepare several problems similar to exercises 7-12 or to have pupils make up problems of their own. Problems ought to include the comparison of two different readings, as in question 10: How long is it between 7:55 a.m. and noon? Similar questions suggested by question 7 are: How many degrees must the temperature rise before the thermometer reads 93°?; how many degrees must it fall before the thermometer reads 37°?; what is the difference between these two readings?

Another series of problems can be based directly on questions 7 through 12. For example, compare each pair of readings listed in question 7:

$68\frac{1}{2}°$ and 67°
$68\frac{1}{2}°$ and $63\frac{1}{2}°$
67° and 50°
$63\frac{1}{2}°$ and 50°

Comparing the four readings in each problem in this way results in six subtraction examples.

On practice sheet 52 we extend the single-digit multiplication algorithm to one three-digit factor. Most of the basic combinations most often missed are included. Again, since examples in the boxes are a summary of the three simpler computations to the left of each, able pupils who can work the examples in the boxes should be excused from the others.

PS-52

a.	b.	c.
3 × 2 = 6	6 × 3 = 18	4 × 5 = 20
3 × 10 = 30	6 × 20 = 120	4 × 20 = 80
3 × 300 = 900	6 × 100 = 600	4 × 300 = 1200
3 × 312 = 936	6 × 123 = 738	4 × 325 = 1300

d.	5 × 5 = 25	20 × 5 = 100	300 × 5 = 1500	325 × 5 = 1625	l.	7 × 9 = 63	00 × 9 = 0	400 × 9 = 3600	407 × 9 = 3663
e.	7 × 2 = 14	10 × 2 = 20	500 × 2 = 1000	517 × 2 = 1034	m.	0 × 7 = 0	90 × 7 = 630	800 × 7 = 5600	890 × 7 = 6230
f.	8 × 5 = 40	40 × 5 = 200	700 × 5 = 3500	748 × 5 = 3740	n.	9 × 6 = 54	40 × 6 = 240	700 × 6 = 4200	749 × 6 = 4494
g.	6 × 4 = 24	40 × 4 = 160	800 × 4 = 3200	846 × 4 = 3384	o.	1 × 8 = 8	90 × 8 = 720	200 × 8 = 1600	291 × 8 = 2328
h.	8 × 6 = 48	50 × 6 = 300	100 × 6 = 600	158 × 6 = 948	p.	4 × 7 = 28	00 × 7 = 0	600 × 7 = 4200	604 × 7 = 4228
i.	9 × 3 = 27	40 × 3 = 120	800 × 3 = 2400	849 × 3 = 2547	q.	7 × 8 = 56	40 × 8 = 320	600 × 8 = 4800	647 × 8 = 5176
j.	6 × 5 = 30	50 × 5 = 250	200 × 5 = 1000	256 × 5 = 1280	r.	0 × 9 = 0	10 × 9 = 90	200 × 9 = 1800	210 × 9 = 1890
k.	8 × 8 = 64	50 × 8 = 400	300 × 8 = 2400	358 × 8 = 2864	s.	9 × 9 = 81	50 × 9 = 450	300 × 9 = 2700	359 × 9 = 3231

Mark correct answer → Mark furthest from correct answer

347 − 152 = 195

(1) 195 (2) 215 (3) 499 (4) 95

(1) ● (2) ○ (3) ⊗ (4) ○ 304 ← Difference between answers you marked

1

28 + 9 + 35 + 14 = 86

(1) 66
(2) 100
(3) 86
(4) 57

(1) ○ (2) ○ (3) ● (4) ⊗ 29

2

324 − 78 = 246

(1) 402
(2) 246
(3) 354
(4) 242

(1) ⊗ (2) ● (3) ○ (4) ○ 156

3

$3.57 × 2 = $7.14

(1) $12.40
(2) $3.59
(3) $1.57
(4) $7.14

(1) ○ (2) ○ (3) ⊗ (4) ● $5.57

4

100 × 17 = 1700

(1) 170
(3) 1700
(2) 107
(4) 710

(1) ○ (2) ⊗ (3) ● (4) ○ 1593

5

18 + 37 + 105 = 160

(1) 520
(3) 155
(2) 382
(4) 160

(1) ⊗ (2) ○ (3) ○ (4) ● 360

6

125 ÷ 25 = 5

(1) 100
(3) 5
(2) 150
(4) 1

(1) ○ (2) ⊗ (3) ● (4) ○ 145

Tickets: Weekdays — Adults..... $1.35 each, Children..... 75¢ each
Weekends — Adults..... $1.65 each, Children..... 85¢ each

7

Mrs. Wilson bought 3 adult tickets for Wednesday night's performance. She paid

(1) $3.35 (2) $2.60 (3) $3.95 (4) $4.05

(1) ○ (2) ⊗ (3) ○ (4) ● $1.45

8

Mr. Henry bought 2 adult and 2 children's tickets for a weekend. He paid

(1) $5.70 (2) $5.00 (3) $7.50 (4) $6.35

(1) ○ (2) ● (3) ⊗ (4) ○ $2.50

9

Mr. Jones decided to buy 7 adult tickets. How much would he save by buying them for a Tuesday rather than for a Saturday?

(1) $2.10 (2) 70¢ (3) $9.45 (4) $5.25

(1) ● (2) ○ (3) ⊗ (4) ○ $7.35

10

A class of 20 children bought tickets for a weekday. Because they went as a group, they paid 10¢ less per ticket than the regular price. How much did their 20 tickets cost?

(1) $12.75 (2) $8.50 (3) $13.00 (4) $15.00

(1) ○ (2) ⊗ (3) ● (4) ○ $4.50

Notes on Pupil Page 53

Tasks: Pupils use the response pattern suggested by the sample at the top of the page to complete the exercises.

Purpose: To test achievement in computational skills with whole numbers.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations.

The Lesson: This pupil page is a continuation of the checkup begun on pupil page 52. The first six examples are typical computation problems. Examples 7 through 10 require the use of such computations in situations involving amounts of money.

Weakness in the computation section calls for basic review and reteaching. Without the computational facility required by examples 1 through 6, a pupil must be stumbling his way along, meeting defeat after defeat. His only hope is to go back to a point at which he can succeed. He needs very special assistance.

The problems involving money are more complex. Computational facility alone is not enough here; pupils must also think about the situation that is presented. You may help pupils understand such problems by discussing the variety of possibilities that can arise. For example, consider all combinations of tickets for weekday performances if no one buys more than four of each kind.

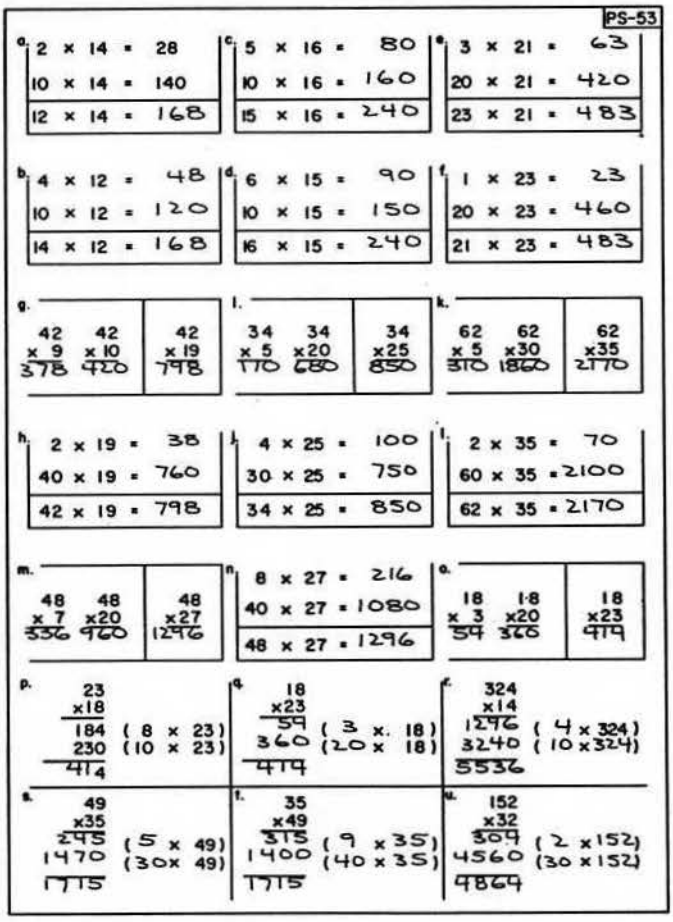
Children's tickets — 75¢ each

Adult's tickets — $1.35 each	0	1	2	3	4
0	0	$ .75	$1.50	$2.25	$3.00
1	$1.35	2.10	2.85	3.60	4.35
2	2.70	3.45	4.20	4.95	5.70
3	4.05	4.80	5.55	6.30	7.05
4	5.40	6.15	6.90	7.65	8.40

We have indicated in black the information you may wish to give the class. The red entries can then be worked out. A table for combinations of weekend ticket sales can be worked out in the same manner.

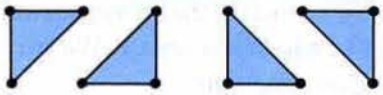
Such detailed analyses help pupils realize that a specific question about a situation is simply one of many questions that could be asked.

Practice sheet 53 concludes the review of the multiplication algorithm. You may wish to check quickly to determine the general level of proficiency each pupil has demonstrated as a guide to possible remedial work.

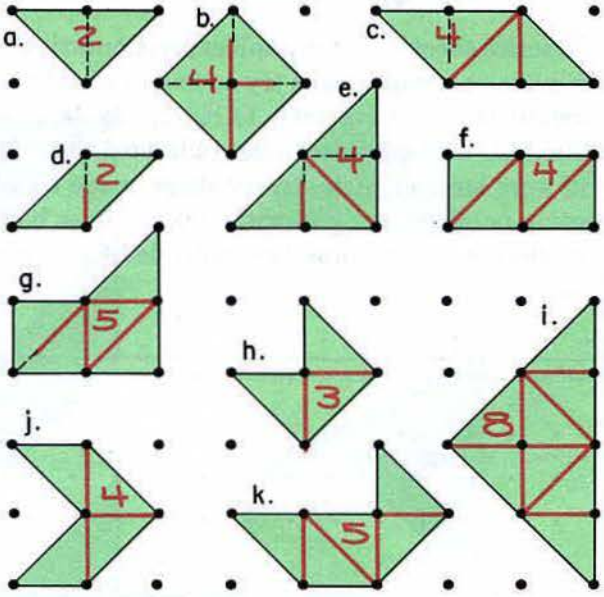


1. How many pieces this size and shape

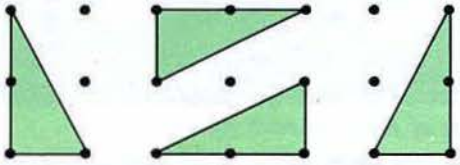
would it take to cover each of the shapes below? The pieces can be turned in any direction.



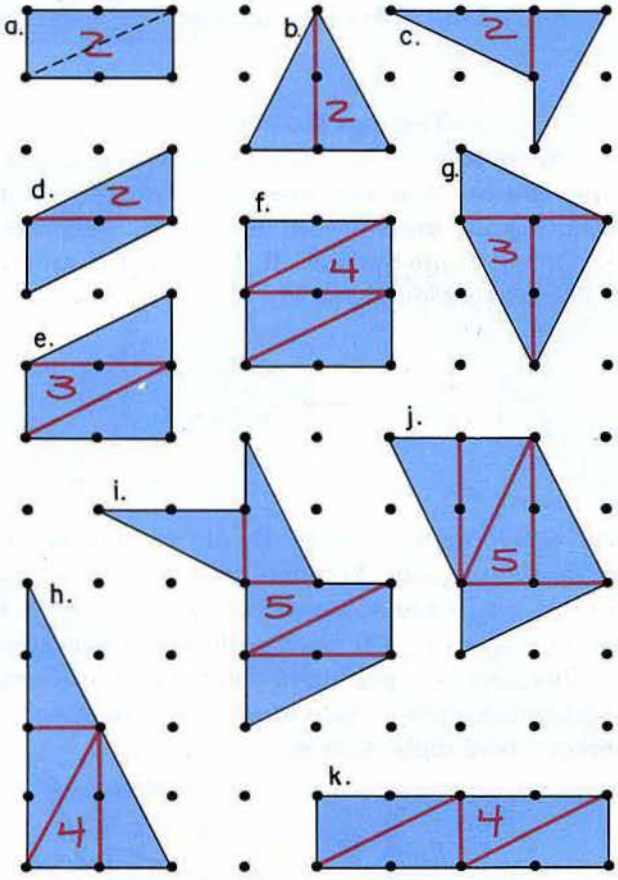
Use lines to show one of the ways you might do it.



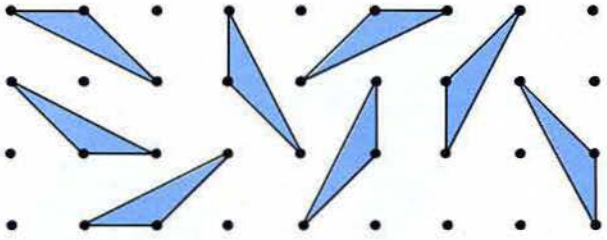
2. Show how many pieces this size and shape



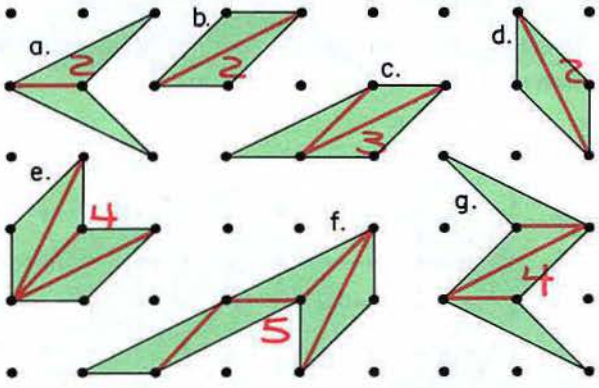
it would take to cover each of the shapes below.



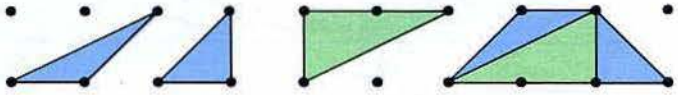
3. How many pieces this size and shape



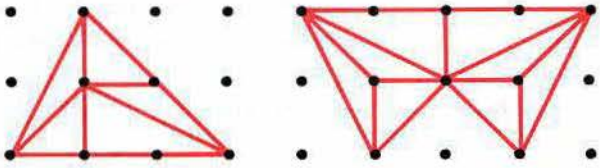
would it take to cover each of the shapes on the right?



4. Make several designs with the three different triangles used above.



OTHER POSSIBILITIES.



54-59 Introducing Some Geometric Ideas

Tasks: *In each of the exercises pupils determine how many specified triangular regions are needed to cover each of the given lettered regions exactly.*

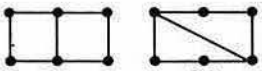
Purpose: To explore the concept of area.

Unifying Ideas: Geometry; measurement.

The Lesson: In the next six pupil pages we explore the idea of area, using various units of area. By convention, we use a square region as a unit in measuring area. However, we could use other shapes as well. For example, on pupil page 54 we use three different triangular regions as units. What different geometric shapes can be built by combining several of these unit regions? We usually pose the question this way: How many of these pieces are required to cover a specific region? Pupils are asked to consider only shapes into which a whole number of the selected unit fits.

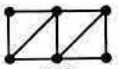
Practice sheet 54 provides practice in comparing the sizes of different shapes. Four shapes are given and the pupil must find a way to divide various regions into a specified combination of these four shapes.

We hope children will notice that two of piece A and two of piece B will cover the same area.

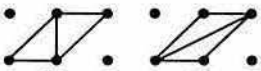


This will help produce the discovery that A and B have the same area.

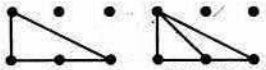
It takes four C's to cover the same area as two A's or two B's.



We notice that two C's cover the same area as two D's.

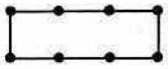


Other relationships may be noticed.



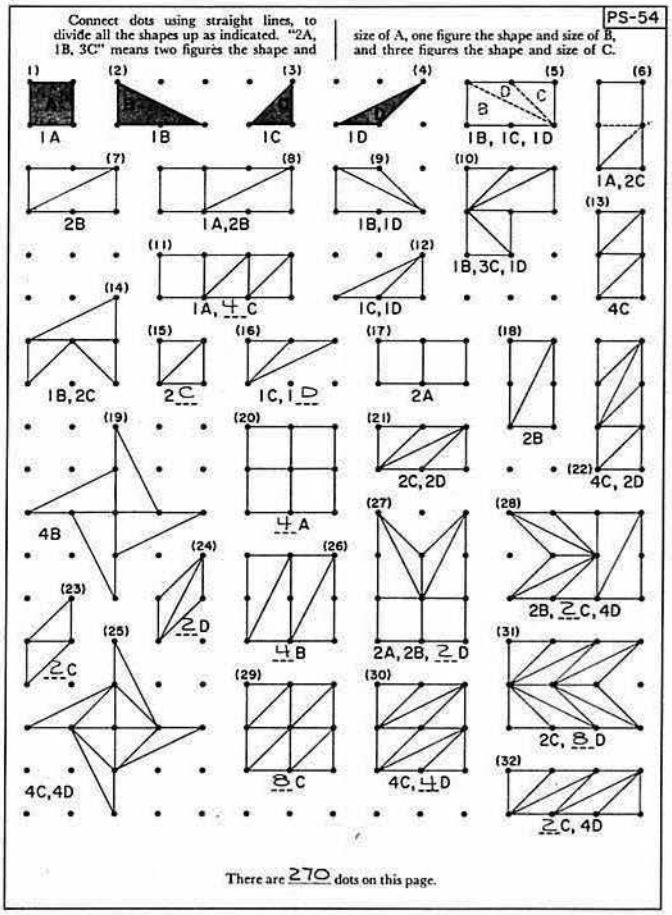
From these investigations, we can conclude that A and B have the same area, C and D have same area, A is twice as large as C or D, and B is twice as large as C or D.

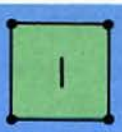
Several shapes on the practice sheet can be divided in several different ways. For example:

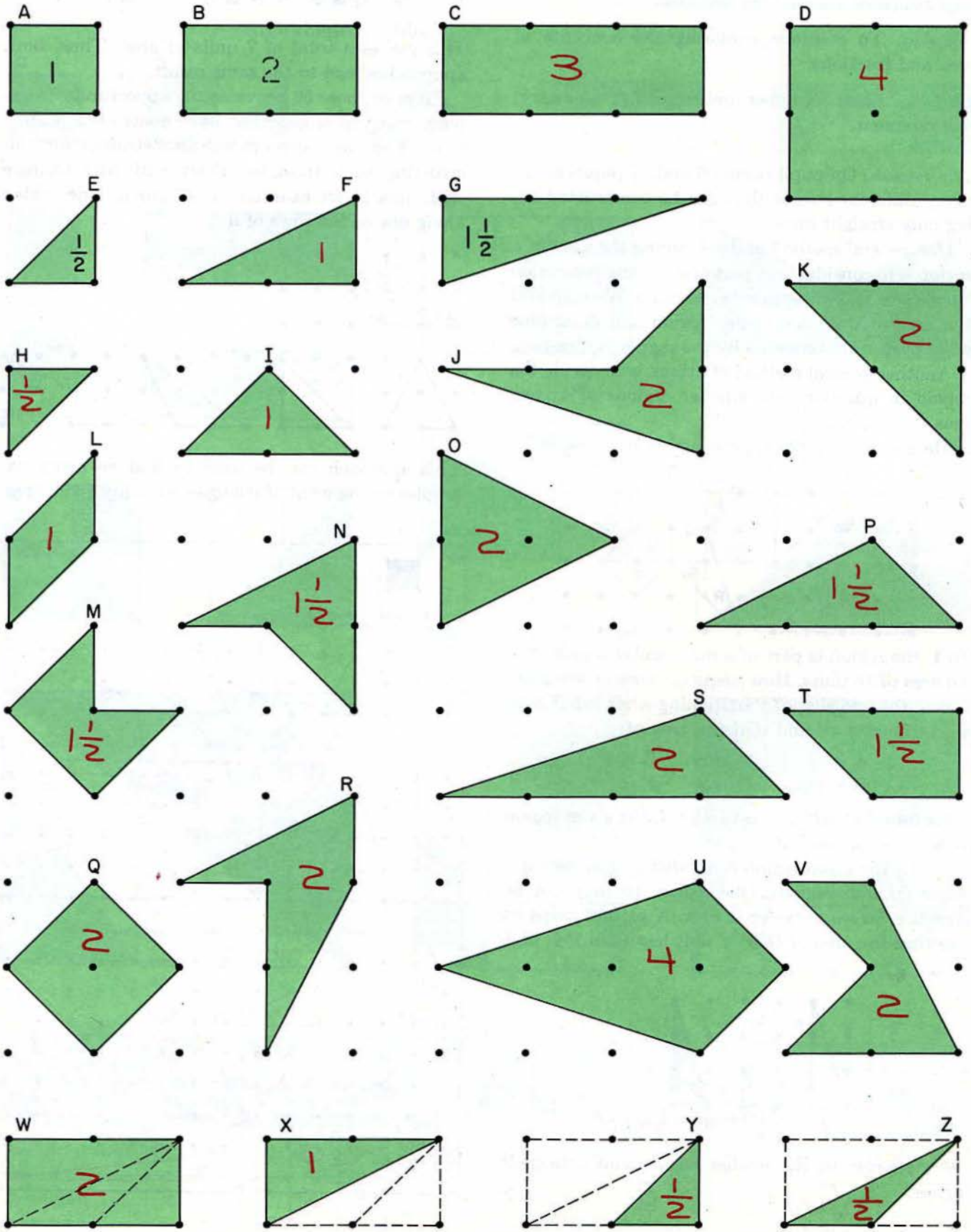


(8) 1A, 2B (11) 1A, 4C

(22) 4C, 2D (32) 2C, 4D



If we consider this green tinted space  as one unit of area,
then how many units of area would each of the tinted spaces below represent?



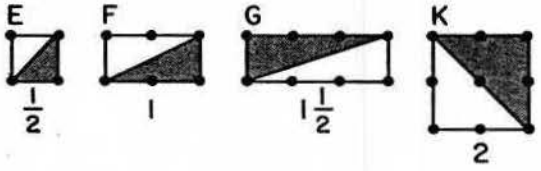
Notes on Pupil Page 55

Tasks: In exercises A-Z pupils determine the number of units of area for each figure.

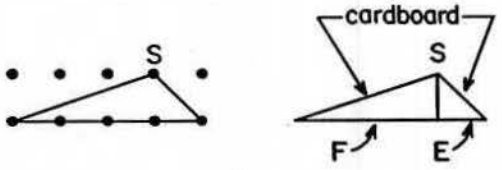
Purpose: To further explore the concept of area and to use it to introduce the concept of fractions.

Unifying Ideas: Number and counting; geometry; measurement.

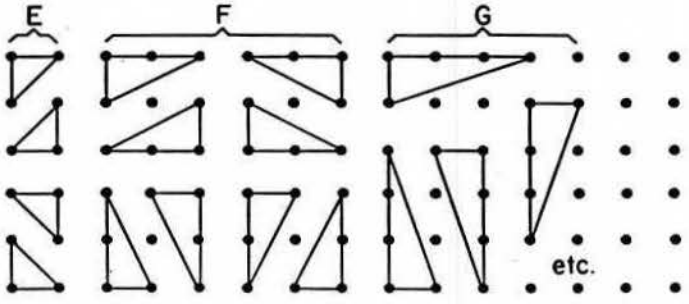
The Lesson: This and the next few pupil pages begin by agreeing on a unit of area. Given a unit of area, we can find the area of a region by simply counting the units needed to cover it. Regions A, B, C, and D are obvious. E, F, G, and K are each half of the region above it.



All other regions (except Z) are combinations of these eight regions. You may wish to cut these eight out of cardboard to construct the regions on the pupil page. Or, you may outline a shape with a rubberband on a peg board and then use other rubberbands to show how to break it up into some combinations of eight shapes.

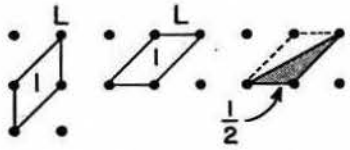


These devices can also be used to help pupils visualize shapes in different positions.

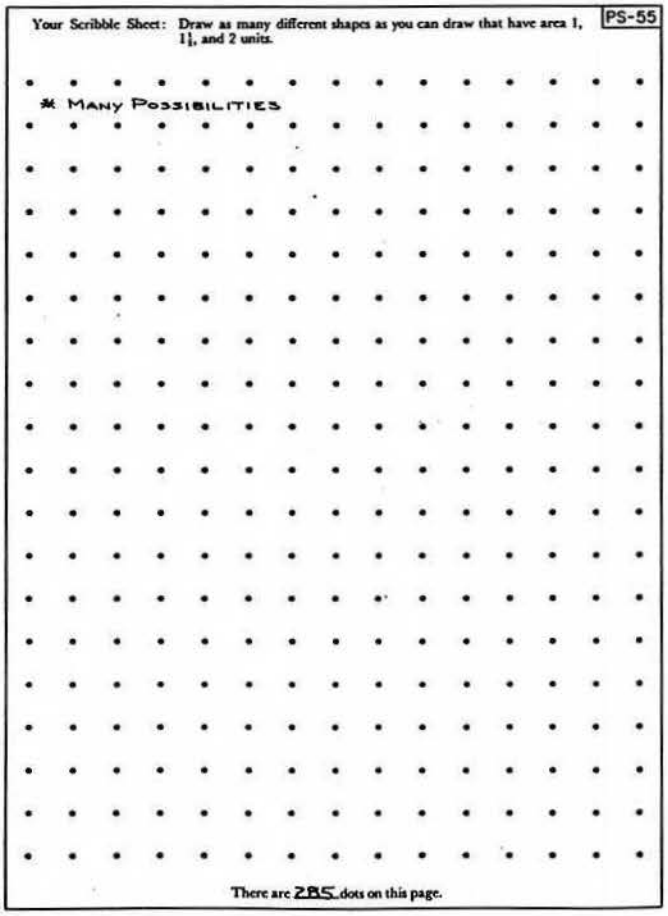


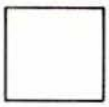
The four regions at the bottom of the pupil page give a preview of the work to come. Note that W, X, and Y are a repetition of B, F, and E. This sequence is repeated so that pupils will recognize that the shaded region in Z is what is left when the shaded region in Y (area ½) is removed from the unshaded region in X (area 1). The shaded region in Z must have area ½.

Another way to reach the same answer is to start with region L, which has area 1. We turn it over and draw its longer diagonal:



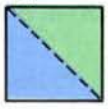
Practice sheet 55 is for pupil experimentation. Individuals or groups might try to find out how many different shapes have area 1, ½, 2, 2½, etc. (An unlimited number of each size could be drawn if the grid were large enough—every shape made by connecting points in the grid with straight lines has an area that is a whole-number multiple of ½ unit of area.)

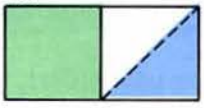

Your Scribble Sheet: Draw as many different shapes as you can draw that have area 1, 1½, and 2 units.
PS-55
* MANY POSSIBILITIES
There are 285 dots on this page.

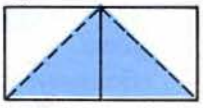
Let us agree that this  is 1 unit of area.

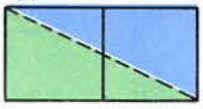
For each of the figures (a-q) below, tell the number of units of area of the

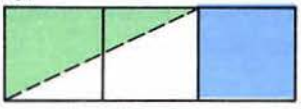
— area shaded in green
— area shaded in blue
— unshaded area
— total area

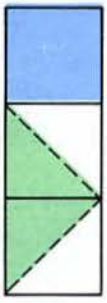
a. 

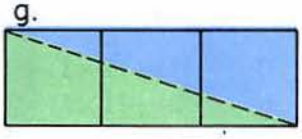
b. 

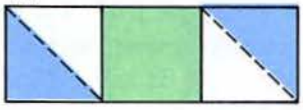
c. 

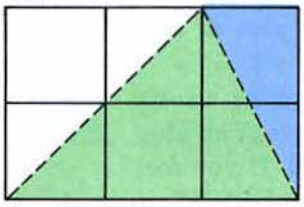
d. 

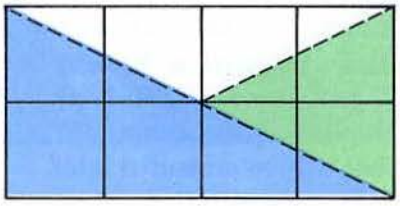
e. 

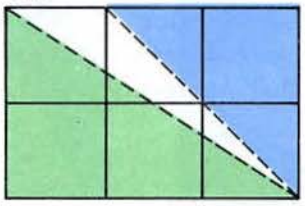
f 

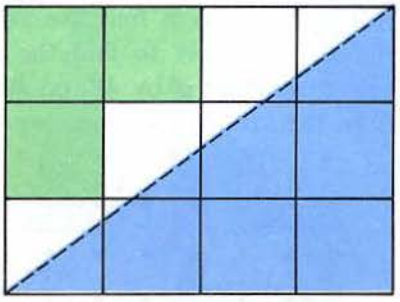
g. 

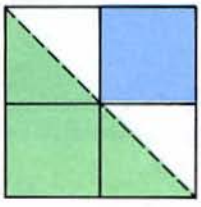
h. 

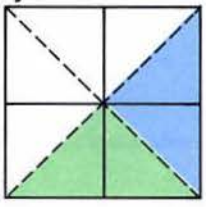
k. 

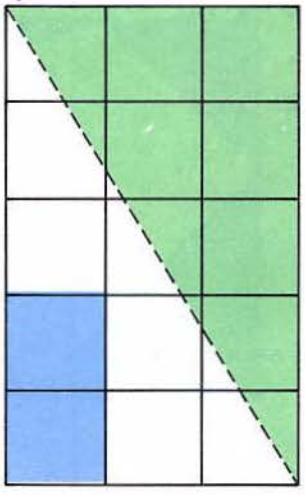
l. 

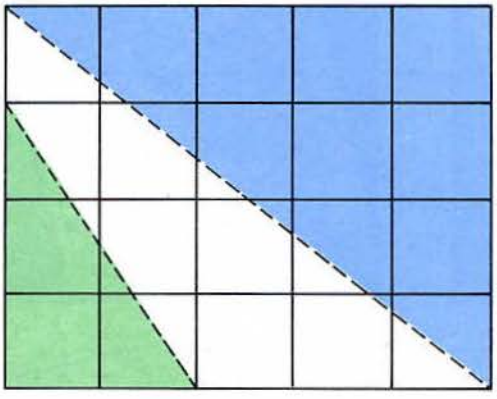
m. 

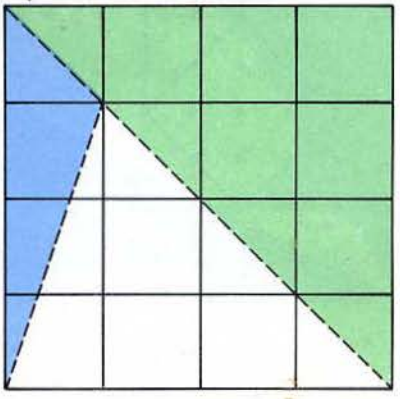
n. 

i. 

j. 

p. 

o. 

q. 

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
area shaded in green	$\frac{1}{2}$	1	0	1	1	1	$1\frac{1}{2}$	1	2	1	3	2	3	3	3	$7\frac{1}{2}$	8
area shaded in blue	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$1\frac{1}{2}$	1	1	1	1	4	2	6	10	2	2
unshaded area	0	$\frac{1}{2}$	1	0	1	1	0	1	1	2	2	2	1	3	7	$5\frac{1}{2}$	6
total area	1	2	2	2	3	3	3	3	4	4	6	8	6	12	20	15	16

Tasks: *Pupils complete the table at the bottom of the page in accordance with the directions.*

Purpose: To continue exploring the concepts of area and fractions.

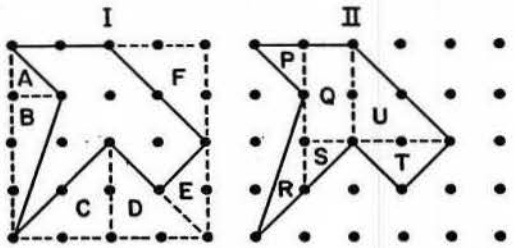
Unifying Ideas: Number and counting; geometry; measurement.

The Lesson: On pupil pages 56 and 57 pupils study many different shapes that can be constructed using only straight lines.

One general method of determining the area of a region is to consider it as part of a rectangular region whose area is a whole number of units. We then find the area of this rectangular region and determine what part of it is covered by the region in question.

Another general method of attack is to divide the region in question into smaller regions of known area.

Here is an example approached in both ways:

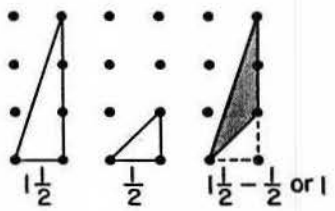


In I, the region is part of a rectangular region with an area of 16 units. How many of those 16 units are not in the odd shape? Partitioning what is left over into triangles we find that the area of:

A is ½	C is 2	E is 1
B is 1½	D is 2	F is 2

This totals 9 units. Since $16 - 9 = 7$, the given region must have an area of 7 units.

In II, the given region is divided up into smaller, more familiar regions. One region, R, may not be familiar, so we consider it first. A picture helps us see that the area of R is ½ unit less than 1½, or 1 unit:

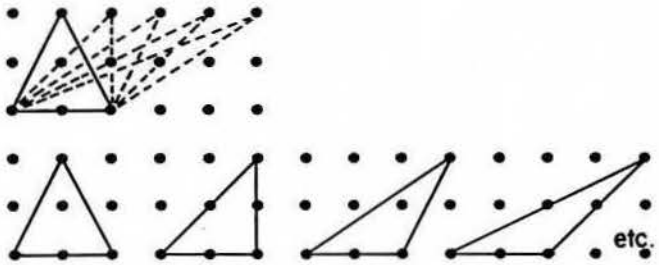


Now we can list the smaller regions and note their areas:

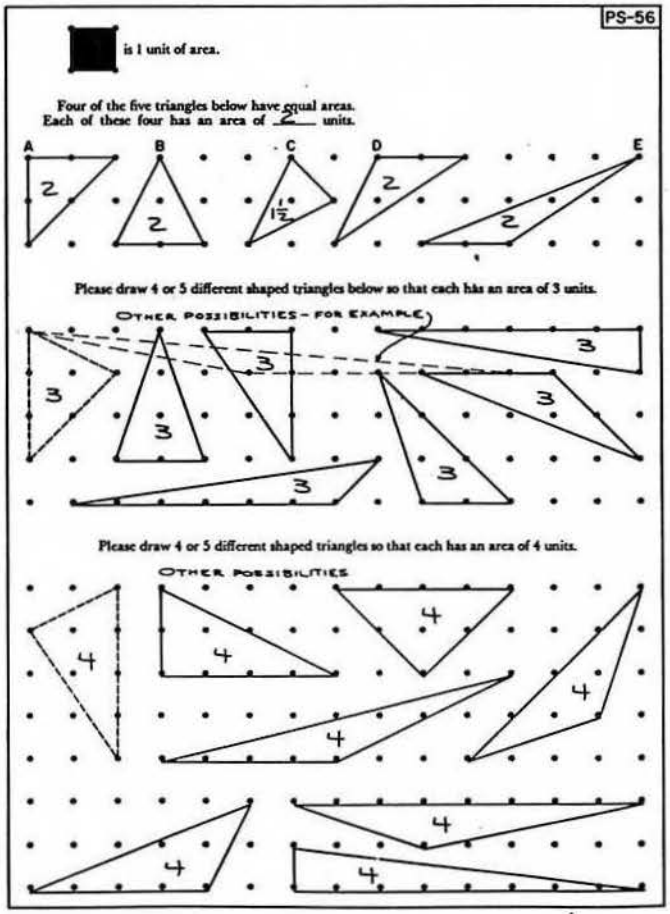
P is ½	R is 1	U is 2
Q is 2	S is ½	T is 1

This makes a total of 7 units of area. Thus, both approaches lead to the same result.

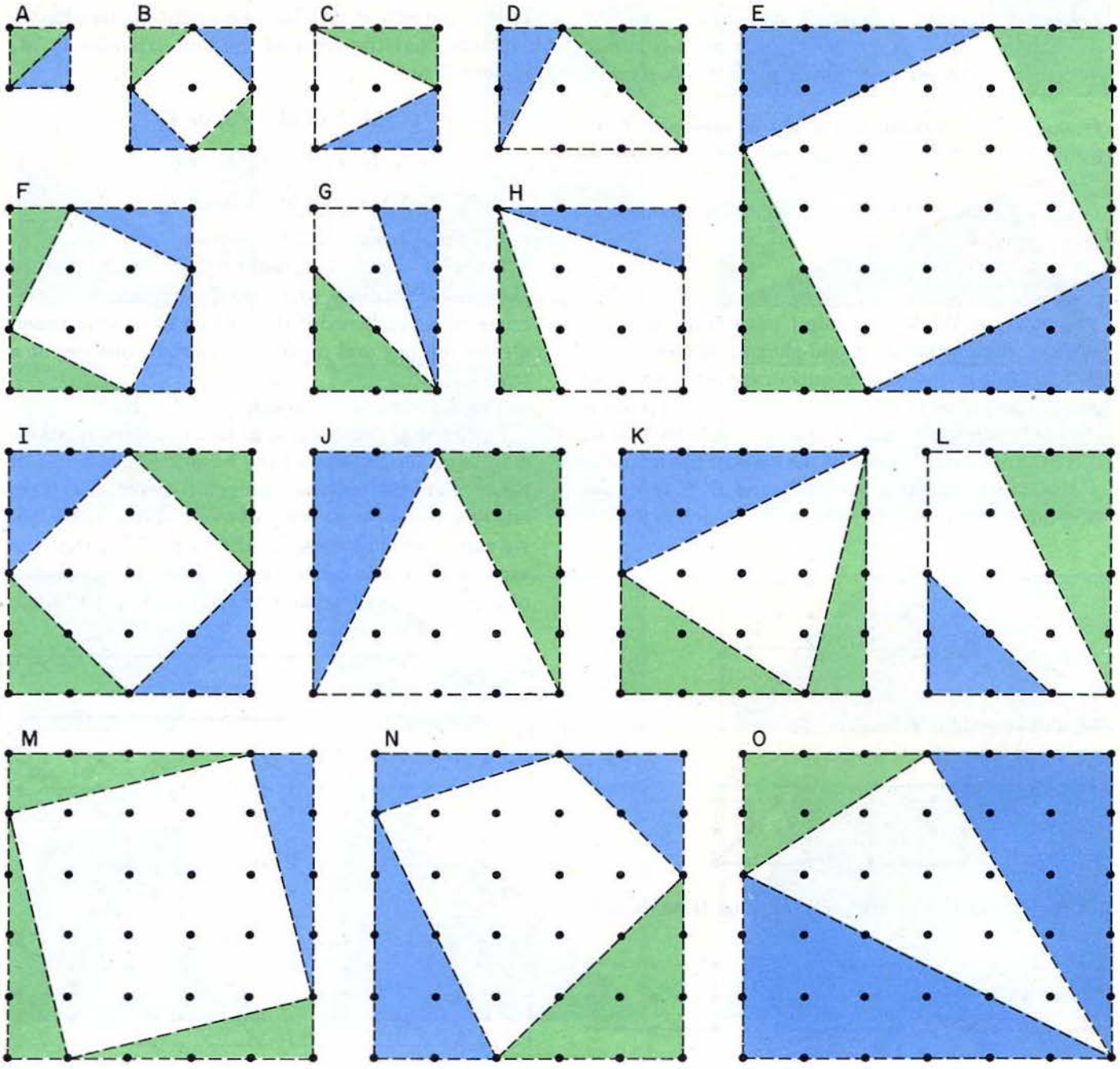
Practice sheet 56 provides the opportunity to explore many triangles that have areas of 2, 3, or 4 units. You might explore a general strategy for constructing such triangles. Start with any triangle with area 2, for example, and "move" one vertex along one of the rows of dots:



This approach can be used to find as many examples as you want of triangles with any given area.


PS-56
is 1 unit of area.
Four of the five triangles below have equal areas.
Each of these four has an area of ___ units.
A
B
C
D
E
Please draw 4 or 5 different shaped triangles below so that each has an area of 3 units.
OTHER POSSIBILITIES - FOR EXAMPLE
Please draw 4 or 5 different shaped triangles so that each has an area of 4 units.
OTHER POSSIBILITIES

On this page we agree that is 1 unit of area.



Units of area	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
shaded in green	$\frac{1}{2}$	1	1	2	8	2	2	$1\frac{1}{2}$	4	4	5	4	6	$4\frac{1}{2}$	3
shaded in blue	$\frac{1}{2}$	1	1	1	8	2	$1\frac{1}{2}$	$1\frac{1}{2}$	4	4	4	2	2	$7\frac{1}{2}$	$16\frac{1}{2}$
unshaded	0	2	2	3	20	5	$2\frac{1}{2}$	6	8	8	7	6	17	13	$10\frac{1}{2}$
total	1	4	4	6	36	9	6	9	16	16	16	12	25	25	30

Tasks: Pupils complete the table.

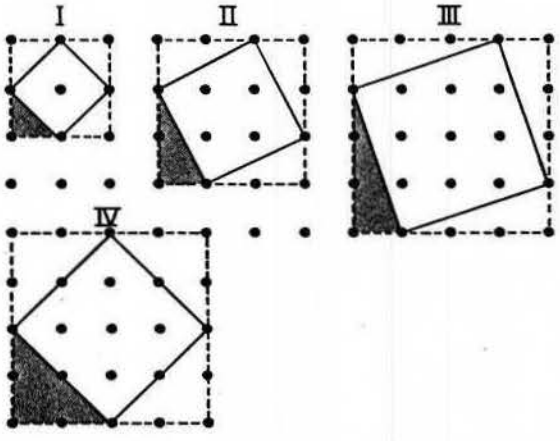
Purpose: To continue studying the concepts of area and fractions.

Unifying Ideas: Number and counting; geometry; measurement.

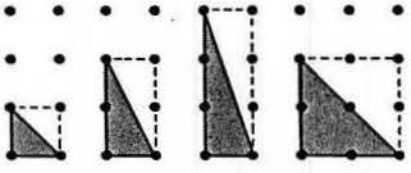
The Lesson: The fact that we can find the area of such shapes as K, N, and O may help strengthen the conviction that the methods we are using are completely general—that we can find the area of any region bounded by straight lines.

Exercises B, E, F, I, and M, as well as all examples on practice sheet 57, are previews of an idea that will be presented later, in Level E. If you feel your class is interested enough, you might ask if anyone can find a shortcut to the area of any square that can be drawn in the grid we have been using.

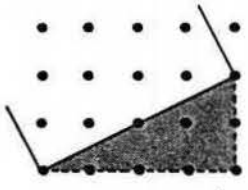
A quick way to find the areas of the following squares is simply to find the area of each corner piece, multiply it by 4, and subtract that product from the area of the larger square.



So all we need look at is one corner of each figure:



In each case, this corner piece is half of a rectangle whose area can be found easily.

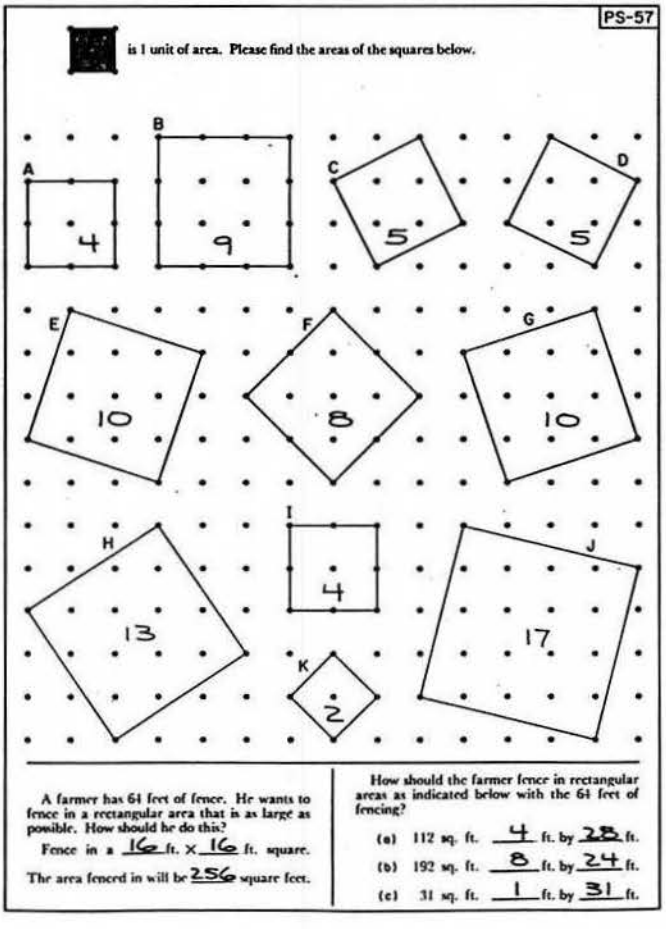


The area of each corner is $(4\times2)\div2=4$. The area of 4 corners, therefore, is $4\times4=16$. From the corner piece we can tell the size of the larger square. Each side must be $4+2=6$, and $6\times6=36$ must be its area. Finally, $36-16=20$, the area of the smaller square.

Someone might even notice that, in this case, the sides of the corner piece that form a right angle are 2 and 4—and $(2\times2)+(4\times4)=20$. To see if this is more than an accident, consider other cases:

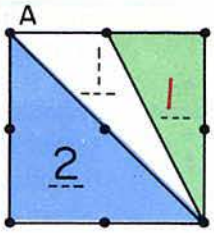
B: $(1\times1)+(1\times1)=2$
F: $(2\times2)+(1\times1)=5$
I: $(2\times2)+(2\times2)=8$
M: $(4\times4)+(1\times1)=17$

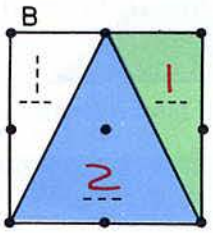
If we consider examples from the practice sheet as well, the conviction grows that this is a reliable shortcut. In fact, it is. It is the Pythagorean Theorem, which states that the square of the hypotenuse of a right triangle can be found by adding the squares of the legs of the triangle.

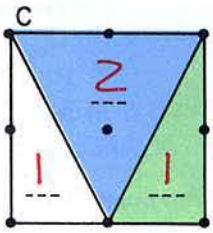

PS-57
is 1 unit of area. Please find the areas of the squares below.
A
4
B
9
C
5
D
5
E
10
F
8
G
10
H
13
I
4
J
17
K
2
A farmer has 64 feet of fence. He wants to fence in a rectangular area that is as large as possible. How should he do this?
Fence in a 16 ft. × 16 ft. square.
The area fenced in will be 256 square feet.
How should the farmer fence in rectangular areas as indicated below with the 64 feet of fencing?
(a) 112 sq. ft. 4 ft. by 28 ft.
(b) 192 sq. ft. 8 ft. by 24 ft.
(c) 31 sq. ft. 1 ft. by 31 ft.

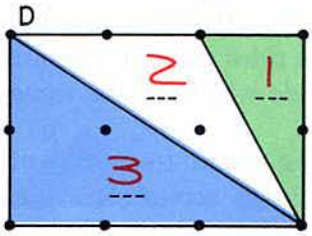
Indicate the size of each blue area first.

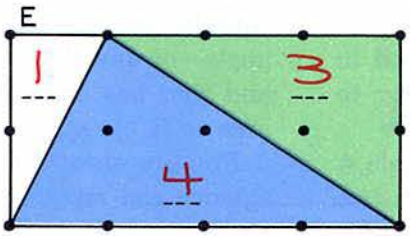
How many blocks this size would be needed to cover the spaces indicated below?

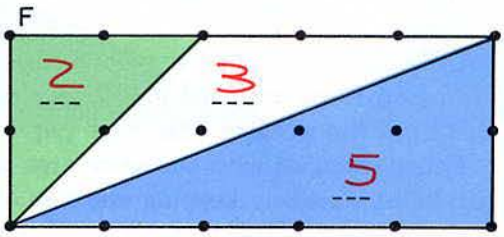


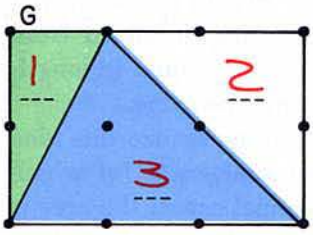


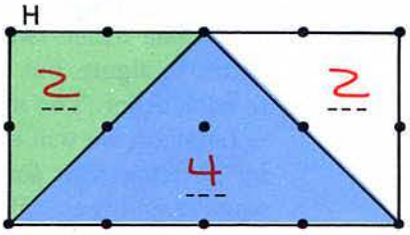


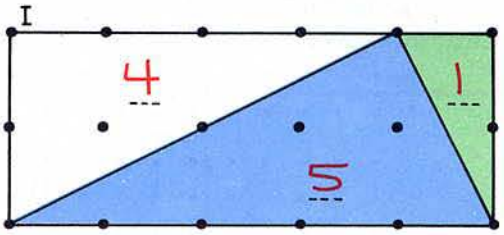


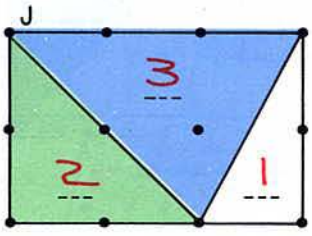


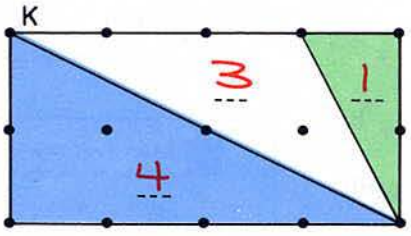


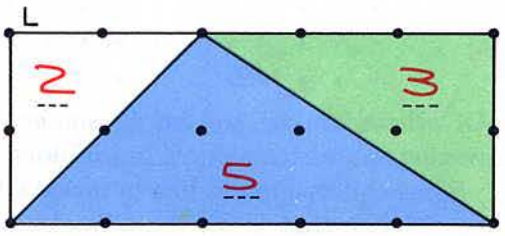


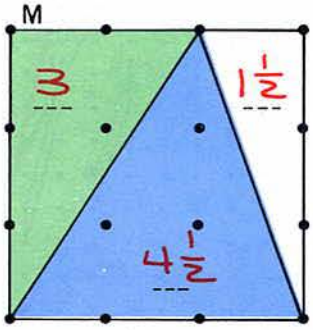


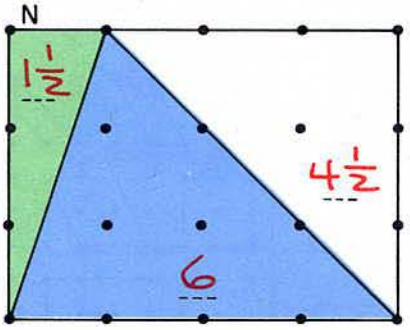


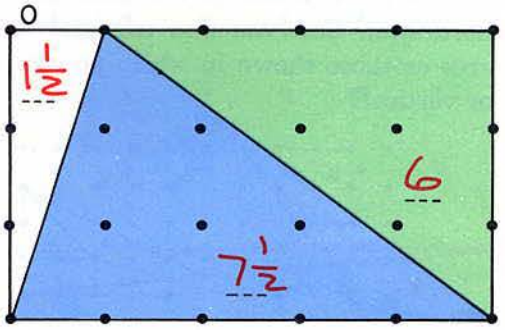




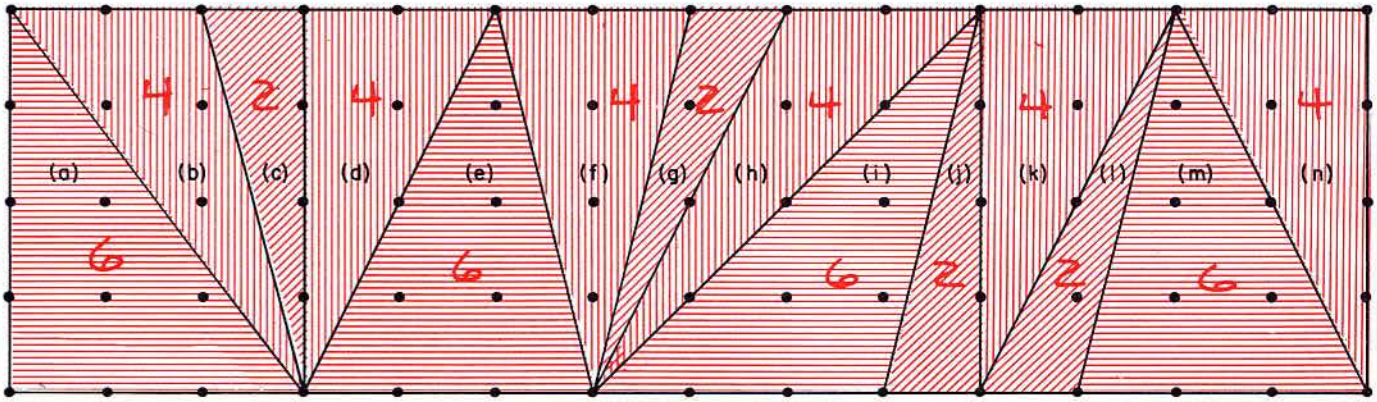








Color all shapes below that are the same size with the same color.



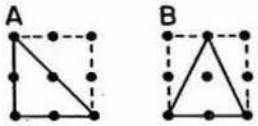
Tasks: In exercises A-O pupils record the number of units of area for each figure. In exercises a-n *pupils follow the directions given for coloring the shapes.*

Purpose: To continue the study of area and fractions.

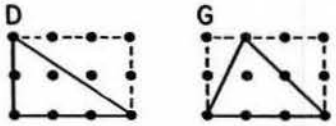
Unifying Ideas: Number and counting; geometry; measurement.

The Lesson: While this pupil page continues the work of finding areas of odd-shaped figures drawn on a grid, it is designed to encourage recognition of general geometric notions—particularly the relationship between the area of a triangle and the area of the rectangle built on one of the sides of the triangle.

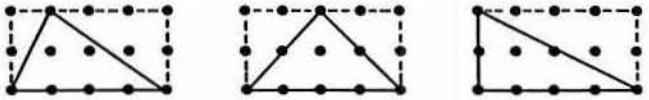
Notice the blue triangles in A and B (C is an upside-down version of B). The area of each is 2:



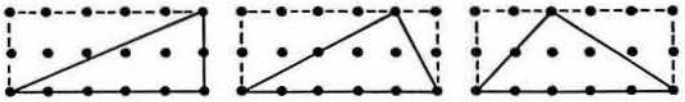
In D and G (J is an upside-down version of G), the area of each blue triangle is 3:



In E, H, and K, the area of each blue triangle is 4:



In F, I, and L, the area of each blue triangle is 5:



We see that:

The area of A and B is half of 2×2.
The area of D, G, and J is half of 2×3.
The area of E, H, and K is half of 2×4.
The area of F, I, and L is half of 2×5.

If this pattern is reliable, we ought to be able to conclude that the areas of the blue triangles in M, N, and O are:

M: half of 3×3, or $4\frac{1}{2}$

N: half of 3×4, or 6

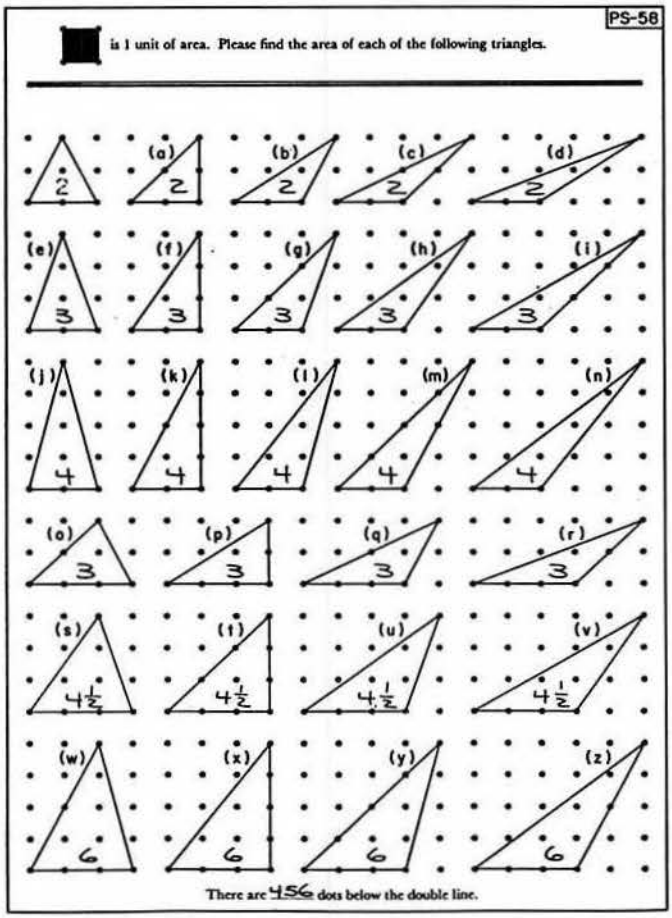
O: half of 3×5, or $7\frac{1}{2}$

All of these results can be verified.

Practice sheet 58 goes further. Each row of sketches illustrates what would happen to a triangle on a peg board if the rubber band were taken off the top peg and moved to the right one peg at a time.

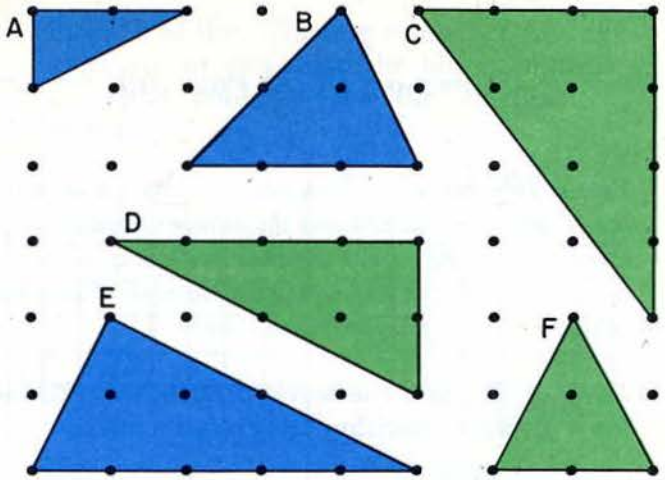
The area remains the same!

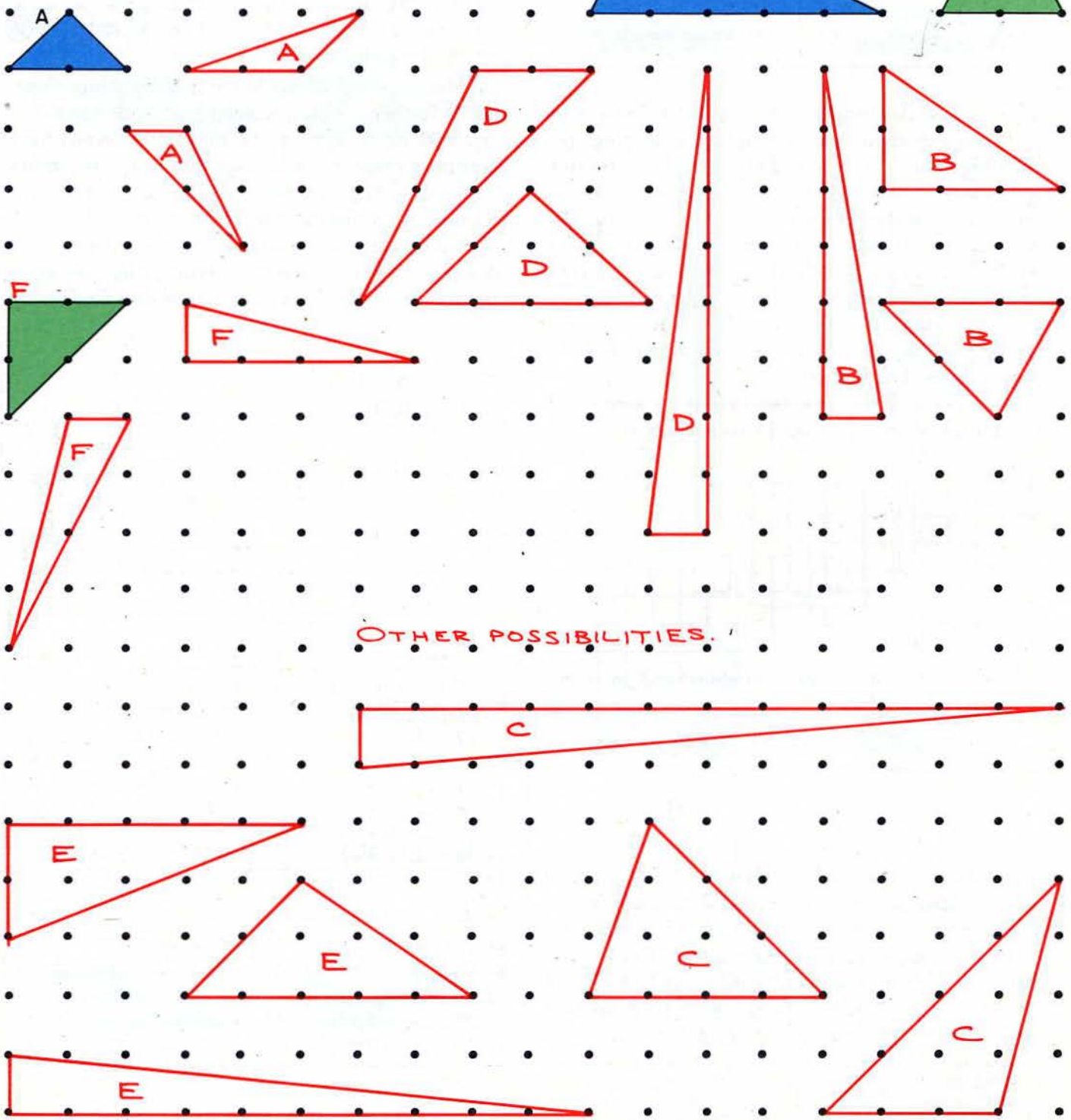
Notice that there seems to be a constant relationship between the size of the base (or "bottom") of these triangles and the distance between that base and the row of pegs we move out along. Later on, we can generalize these results by noticing that the area of a triangle is half the product of its base and altitude. You may wish to mention this in passing.



By connecting dots as in the examples given, draw at least three more triangles the same size as each example. Make all triangles different in shape. Two triangles are the same shape if one can be cut out and can be placed so that it exactly fits on the other. Otherwise they are of different shapes.

Label all triangles the same size with the letter shown on the right.





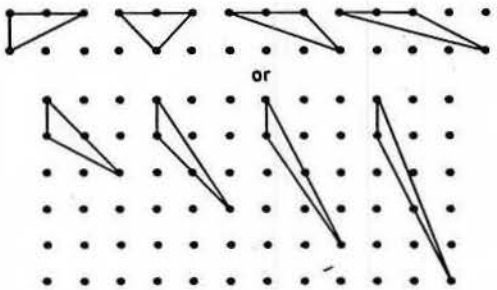
Tasks: *Pupils draw and label triangles in accordance with the directions.*

Purpose: To continue studying area concepts.

Unifying Ideas: Geometry; measurement.

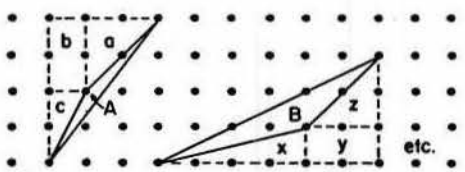
The Lesson: The peg board activity described in Notes on Pupil Page 58 provides a quick way to carry out the work on this pupil page.

Other triangles with the same area as triangle A can be obtained by keeping one side of length two and keeping the third vertex one row away or by keeping one side of length one and the third vertex two rows away.



All others can be treated in similar fashion—by moving one vertex along a line of dots.

Some children might like to tackle a harder problem. Notice that in all the triangles we have considered there is always one side that is vertical or horizontal. Can you find triangles with the same area as those shown in which no side is horizontal or vertical?



Area of:

$$A+a+b+c=\frac{1}{2}\text{ of }12=6$$

$$A+2+2+1=6$$

$$\text{Area of A}=1$$

Area of:

$$B+x+y+z=\frac{1}{2}\text{ of }18=9$$

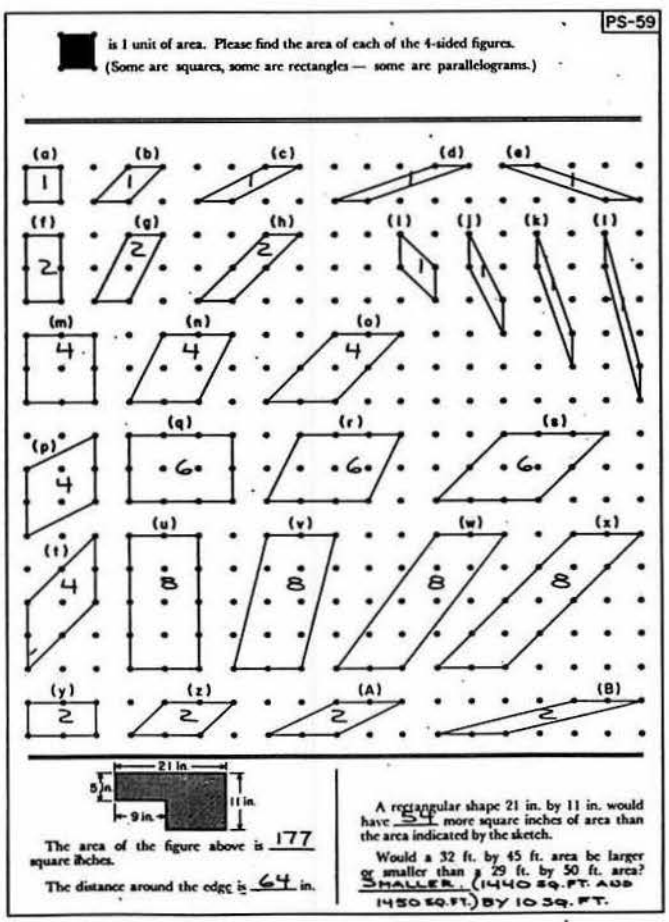
$$B+2+2+2=9$$

$$\text{Area of B}=3$$

In this search, pupils will have rich experiences and may come up with surprising conclusions. (For example, the sides of A, as given, contain 4 points and there are no points inside A; pupils may conclude, correctly, that *any* triangle whose sides contain exactly 4 points and that has none inside will have the same area as A. Similarly, the sides of B contain 6 points and B has 1 point inside. Any triangle—or any shape—whose sides contain 6 points and that has one point inside will have the same area as B.

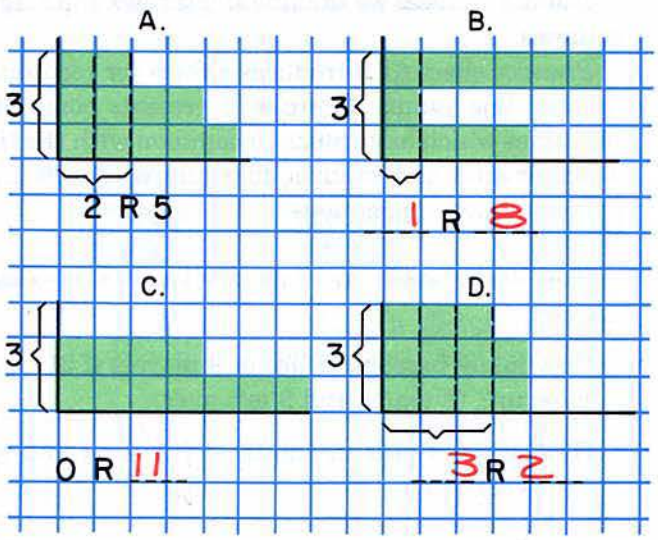
Practice sheet 59 is a peg board treatment of squares and rectangles. As we go across each row, the base of all figures is the same length. The opposite side remains the same length, but is moved over. The other two sides change in length from figure to figure but always remain equal in length to each other. The area remains the same.

Later on we will expand and generalize this idea by noticing that for any parallelogram (of which squares and rectangles are special cases) the area is the product of the base and the altitude, regardless of the shape.


PS-59
is 1 unit of area. Please find the area of each of the 4-sided figures.
(Some are squares, some are rectangles — some are parallelograms.)
21 in.
5 in.
11 in.
9 in.
The area of the figure above is 177 square inches.
The distance around the edge is 64 in.
A rectangular shape 21 in. by 11 in. would have 54 more square inches of area than the area indicated by the sketch.
Would a 32 ft. by 45 ft. area be larger or smaller than a 29 ft. by 50 ft. area?
SMALLER. (1440 SQ. FT. AND 1450 SQ. FT.) BY 10 SQ. FT.

Arranging Tiles in Rows and Columns.

Here are several ways of arranging 11 tiles in 3 rows, and a shorthand method of recording these arrangements.



Arrangement D has more complete columns than the others. In other words, the "remainder" has the fewest tiles.

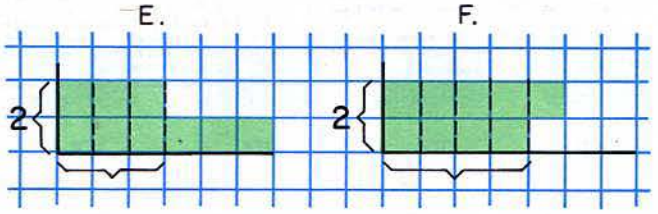
So, if we want the "most complete columns" or the "smallest remainder," only arrangement D would be a solution.

Here is a shorthand way to state a problem about arranging tiles:

$$2\overline{)9}$$

The shorthand states the question: What is the largest number of full columns (or the smallest remainder) if 9 tiles are arranged in 2 rows?

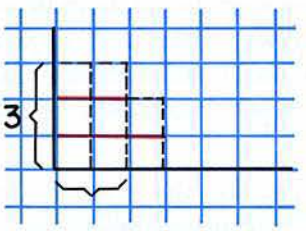
While both sketches below show 9 tiles arranged in 2 rows, only one has the most full columns (or the smallest remainder).



3 R 3 4 R 1

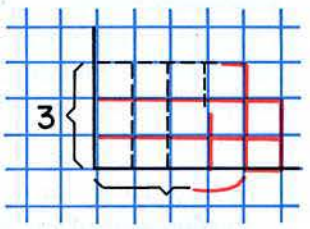
For each of the following examples, find the arrangement of tiles with the largest number of complete columns (or the smallest remainder).

G. $3\overline{)8}$



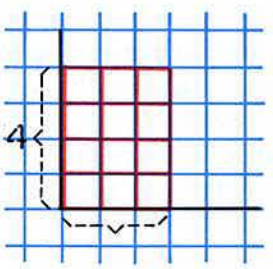
2 R 2

H. $3\overline{)14}$



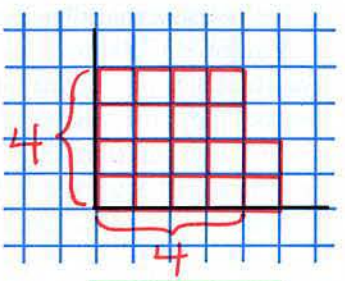
4 R 2

I. $4\overline{)12}$



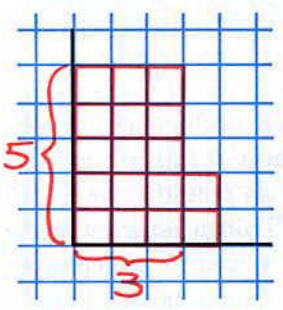
3 R 0

J. $4\overline{)18}$



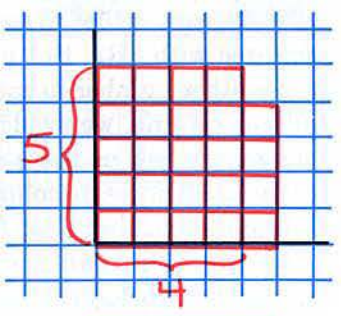
4 R 2

K. $5\overline{)17}$



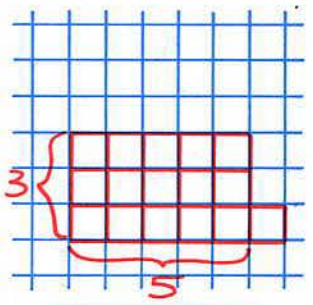
3 R 2

L. $5\overline{)24}$



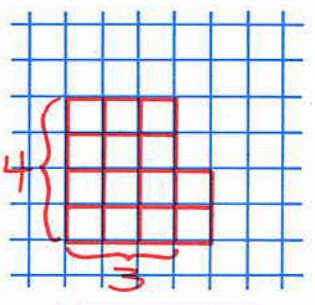
4 R 4

M. $3\overline{)16}$



5 R 1

N. $4\overline{)14}$



3 R 2

60-63 Exploring the idea through arrangements of tiles in columns and rows

Tasks: *In exercises A-F pupils are to record the number of complete columns and the number remaining. In exercises G-N they draw sketches and complete records for arrangements with the largest number of complete columns (or the smallest remainder).*

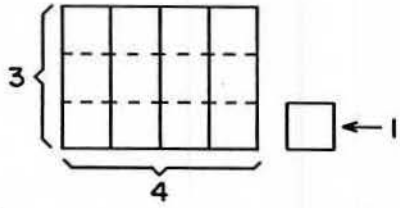
Purpose: To begin the development of the rationale for a division algorithm for whole numbers.

Unifying Ideas: Multiplication and division.

The Lesson: Arithmetic is essentially a method of reporting experience obtained by manipulating and counting things. How would you report an attempt to arrange 13 tiles in a rectangular array with columns of 3 tiles each? (By a rectangular array of tiles we mean an arrangement in rows and columns in which all columns contain the same number of tiles and all rows contain the same number of tiles.)

Is it enough to say "impossible"? For some purposes, perhaps. If, on the other hand, we wish to convey more information, we can say:

I can place 12 tiles in columns of 3, making 4 full columns, and have 1 of the 13 tiles left over.



The results could be noted in shorthand in many ways:

$$13 = 3 \times 4 + 1 \qquad 13 = 4 \times 3 + 1$$

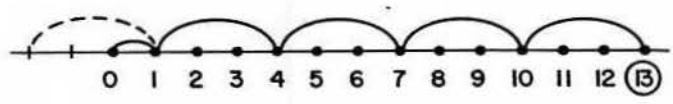
$$3\overline{)13} \qquad 3\overline{)13}$$

$$4\text{R}1 \qquad 4\text{R}1$$

with "R" as shorthand for "Remainder."

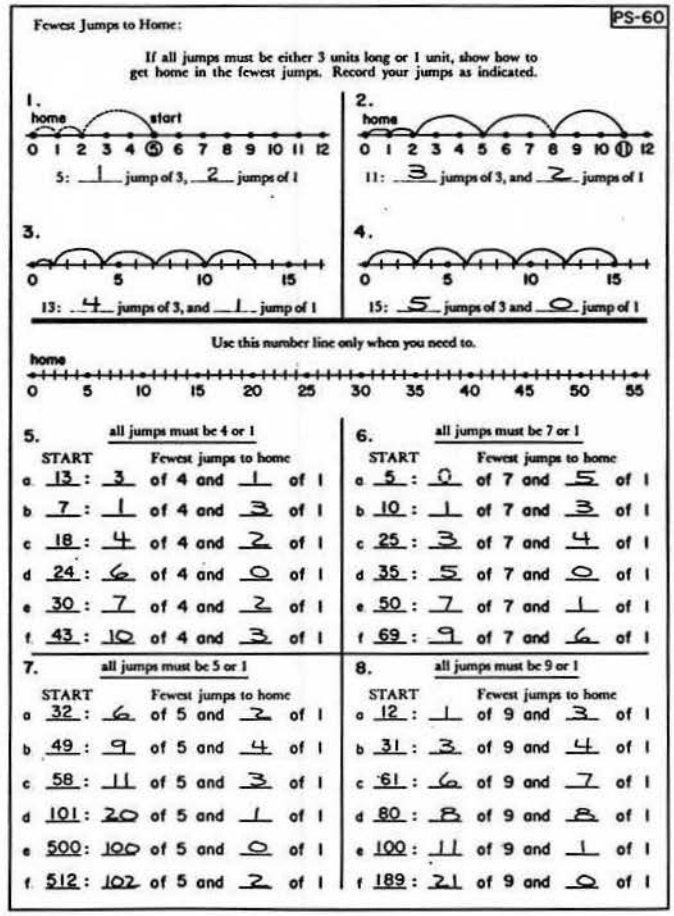
Pupil page 60 introduces one of the methods of recording such results of experiments.

Practice sheet 60 presents another kind of experiment that may seem very different at first. If I am at 13 on a number line and wish to jump to 0 in jumps of 3 at a time, how could I do it? Again, "impossible" is one answer that conveys an important piece of information. But we can say more. We could say that the most full jumps of 3 I could take would be 4. We can add to that information by saying that after 4 jumps of 3, we could complete the journey with a jump of 1. We can draw a sketch of our results:



On practice sheet 60 we do not introduce any shorthand. Rather, we simply keep a running summary, not wishing to burden the pupil with symbolism. Nor do we wish to point too soon to the similarity between arranging tiles in rectangular arrays and jumping on a number line

Pupil pages and practice sheets will be used to develop these two themes independently. We bring them together in practice sheets 64 and 65.



Fewest Jumps to Home: PS-60

If all jumps must be either 3 units long or 1 unit, show how to get home in the fewest jumps. Record your jumps as indicated.

1. home, start — 0 1 2 3 4 5 6 7 8 9 10 11 12 — 5: 1 jump of 3, 2 jumps of 1

2. home — 0 1 2 3 4 5 6 7 8 9 10 11 12 — 11: 3 jumps of 3, and 2 jumps of 1

3. 0 5 10 15 — 13: 4 jumps of 3, and 1 jump of 1

4. 0 5 10 15 — 15: 5 jumps of 3 and 0 jump of 1

Use this number line only when you need to.

home — 0 5 10 15 20 25 30 35 40 45 50 55

5. all jumps must be 4 or 1

START	Fewest jumps to home
a 13 :	3 of 4 and 1 of 1
b 7 :	1 of 4 and 3 of 1
c 18 :	4 of 4 and 2 of 1
d 24 :	6 of 4 and 0 of 1
e 30 :	7 of 4 and 2 of 1
f 43 :	10 of 4 and 3 of 1

6. all jumps must be 7 or 1

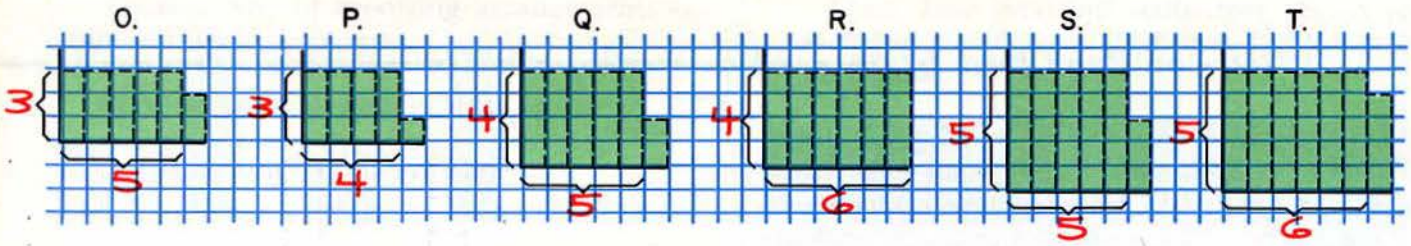
START	Fewest jumps to home
a 5 :	0 of 7 and 5 of 1
b 10 :	1 of 7 and 3 of 1
c 25 :	3 of 7 and 4 of 1
d 35 :	5 of 7 and 0 of 1
e 50 :	7 of 7 and 1 of 1
f 69 :	9 of 7 and 6 of 1

7. all jumps must be 5 or 1

START	Fewest jumps to home
a 32 :	6 of 5 and 2 of 1
b 49 :	9 of 5 and 4 of 1
c 58 :	11 of 5 and 3 of 1
d 101 :	20 of 5 and 1 of 1
e 500 :	100 of 5 and 0 of 1
f 512 :	102 of 5 and 2 of 1

8. all jumps must be 9 or 1

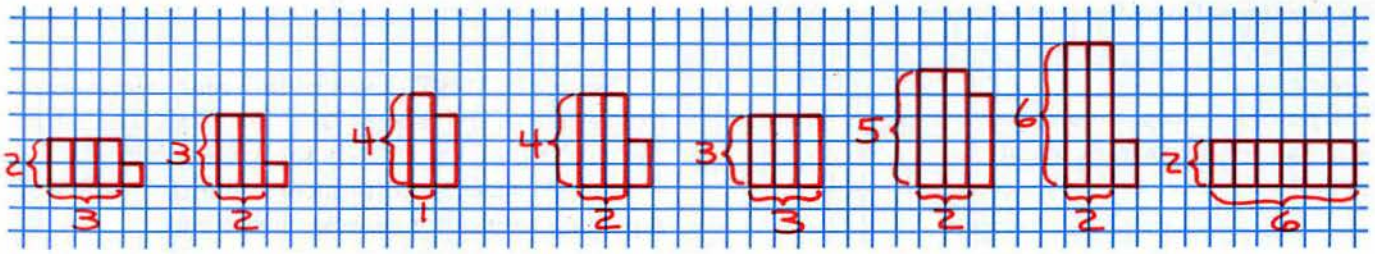
START	Fewest jumps to home
a 12 :	1 of 9 and 3 of 1
b 31 :	3 of 9 and 4 of 1
c 61 :	6 of 9 and 7 of 1
d 80 :	8 of 9 and 8 of 1
e 100 :	11 of 9 and 1 of 1
f 189 :	21 of 9 and 0 of 1



Describe these arrangements in shorthand.

O.	P.	Q.	R.	S.	T.
3|17	3|13	4|22	4|24	5|28	5|34
5 R 2	4 R 1	5 R 2	6 R 0	5 R 3	6 R 4

Make a sketch of each of the arrangements suggested below and complete the shorthand for each arrangement.



a.	b.	c.	d.	e.	f.	g.	h.
2|7	3|7	4|7	4|10	3|9	5|14	6|14	2|12
3 R 1	2 R 1	1 R 3	2 R 2	3 R 0	2 R 4	2 R 2	6 R 0

Complete the shorthand below.
Draw sketches only if you need to.

I	II	III	IV
3|25	4|25	5|25	6|25
8 R 1	6 R 1	5 R 0	4 R 1

V	VI	VII	VIII
4|14	4|20	4|30	9|30
3 R 2	5 R 0	7 R 2	3 R 3

IX	X	XI	XII
5|17	6|30	9|23	5|32
3 R 2	5 R 0	2 R 5	6 R 2

XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX
5|26	4|43	6|39	8|31	7|40	10|100	7|50	1|1
5 R 1	10 R 3	6 R 3	3 R 7	5 R 5	10 R 0	7 R 1	1 R 0

Tasks: *In exercises O-T pupils use the shorthand notation to describe the tile arrangements. In exercises* a-h *pupils sketch arrangements of tiles and complete the shorthand notation for each. In exercises I-XX pupils complete the shorthand descriptions of tile arrangements.*

Purpose: To continue to develop the rationale for a division algorithm for whole numbers.

Unifying Ideas: Multiplication and division.

The Lesson: Our experiments and reporting results continue. Pupils sketch arrangements of tiles and complete shorthand reports.

You might like to raise a question concerning reconstruction of a report if some of the results were erased. In how many of the following reports is enough information left to complete the report? Remember that the remainder is the number of tiles in less than a full column.

(a) $3\overline{)17}$ R

(b) $\overline{)20}$ 6R2

(c) $\overline{)14}$ 3R

Enough is given in *a*: 5R2. Enough is given in *b*: 3 tiles in a column. Example *c* requires some thought. If there were 4 tiles in each column, we could finish this one with 3R2. But maybe there were 3 or 5 or some other number in each column. If there were 3 in each column, we would have 4 full columns—but our report tells us there are only 3 full columns. If there are 5 in each column, we would have 2 full columns—but that's too few. Things get even worse if we try fewer than 3 or more than 5 per column. So we can conclude there must have been 4 in each column and a remainder of 2.

How many of the following provide enough information?

(d) $6\overline{)}$ 5R3

(e) $.9\overline{)69}$ R

(f) $\overline{)25}$ RO

(g) $\overline{)50}$ R1

(h) $\overline{)10}$ R1

(i) $13\overline{)}$ 6R2

There is enough in *d*, *e*, and *i*, but not enough in *f*, *g*, and *h*—unless we exclude single rows and single columns.

Practice sheet 61 introduces a form for recording number line jumps. Exercise 4 presents some applications which have much in common with the tile arrangements and number line jumps. Pupils can propose other applications:

How many weeks in 10 days? (1 and 3 days left over)

How many baseball teams of 9 players if 21 boys show up? (2 teams and 3 left over)

How could 3 girls divide 20¢? (6¢ each and 2¢ left over)

Fewest Jumps to Home: PS-61

Here is a way of recording examples of "fewest jumps home."

1.

	A	B	C	D	E	F	G	H	I
From	17	28	28	28	28	57	108	108	322
Size of large jumps	5	4	3	6	8	10	25	5	3
No. of large jumps	3	7	9	4	3	5	4	21	107
Remainder (No. of 1 jumps left to complete the trip to home.)	2	0	1	4	4	7	8	3	1

2.

	J	K	L	M	N	O	P	Q	R	S	T	U
F	14	22	36	45	32	77	30	50	80	50	40	90
S	3	3	3	7	7	7	9	9	9	12	12	12
N	4	7	12	6	4	11	3	5	8	4	3	7
R	2	1	0	3	4	0	3	5	8	2	4	6

3.

	V	W	X	Y	Z	a	b	c	d	e	f	g
F	30	7	37	40	15	55	100	49	149	200	37	237
S	3	3	3	2	2	2	5	5	5	10	10	10
N	10	2	12	20	7	27	20	9	29	20	3	23
R	0	1	1	0	1	1	0	4	4	0	7	7

4. What Can You Say?

Mary had 26 books to arrange on 3 shelves. She wanted the number of books on each shelf to be the same, but that is impossible.

She could put at least 8 on each shelf and have 2 left over. She wanted two shelves to be alike. She could PUT THE 2 REMAINING BOOKS ON ONE SHELF OR ONE ON EACH OF TWO SHELVES.

Mrs. Ross collected 102 eggs from the hen house. She put them in 1-dozen egg cartons. She wanted as many full cartons as possible.

She could MAKE 8 FULL CARTONS AND HAVE 6 EGGS LEFT OVER

Mr. Ross brought in 81 eggs from the other hen houses. All together, with eggs from both hen houses, they could fill 15 cartons and have 3 eggs left over.

Another way of reporting arrangements of tiles in rows and columns.

Consider the following problems:

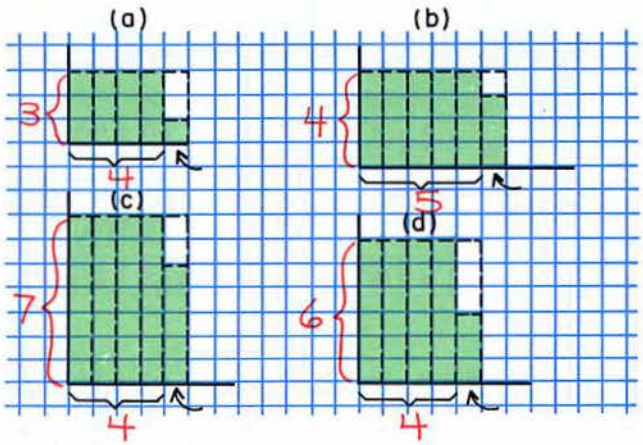
(a) arrange 13 tiles in 3 rows.

(b) arrange 23 tiles in 4 rows.

(c) arrange 33 tiles in 7 rows.

(d) arrange 27 tiles in 6 rows.

Sketches of the arrangements might look like this:



We can easily report the number of full columns and the remainders:

(a) 4R1 (b) 5R3

(c) 4R5 (d) 4R3

This method of reporting includes a record of the number of tiles in the incomplete column.

Another method of reporting tells us something else about the incomplete column. It tells us what part of the column is filled by the remaining tiles. Here is a way to describe this relation:

(a) $\frac{1}{3}$ (1 out of 3 or one-third full)

(b) $\frac{3}{4}$ (3 out of 4 or three-fourths full)

(c) $\frac{5}{7}$ (5 out of 7 or five-sevenths full)

(d) $\frac{3}{6}$ (3 out of 6 or THREE-SIXTHS full)

These four problems with their results reported in this new way are usually written in this fashion:

(a)	(b)	(c)	(d)
$3\overline{)13}$ $4\frac{1}{3}$	$4\overline{)23}$ $5\frac{3}{4}$	$7\overline{)33}$ $4\frac{5}{7}$	$6\overline{)27}$ $4\frac{3}{6}$

This new way of reporting tells us something about arranging tiles in columns of a given size. But we would write the same reports if we were arranging marbles or cakes or pennies in groups of a given size. So we can read these reports in a general way:

(a) Thirteen divided by three equals four and one-third.

(b) Twenty-three divided by four equals five and three-fourths.

Look at example (d). We could have described the incomplete column (or group) in two ways, namely:

R3 (old way)

or:

$\frac{3}{6}$ or $\frac{1}{2}$ (new way)

since the incomplete column (or group)

has 3 tiles (or things) in it,

or

is three-sixths or one-half full.

Please finish the following reports in the new shorthand. (In C and D report 14 divided by 4 in different ways.)

A	B	C	D
$3\overline{)17}$ $5\frac{2}{3}$	$6\overline{)35}$ $5\frac{5}{6}$	$4\overline{)14}$ $3\frac{2}{4}$	$4\overline{)14}$ $3\frac{1}{2}$

or

Notes on Pupil Page 62

Tasks: *In exercises* a-d *and A-D pupils adopt a new shorthand notation for recording given arrangements of tiles.*

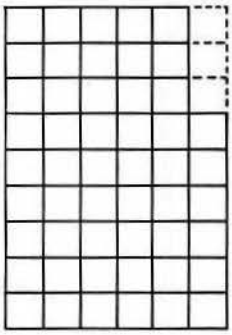
Purpose: To introduce pupils to a conventional notation used to describe results of division for whole numbers.

Unifying Ideas: Numeration; multiplication and division.

The Lesson: Here we introduce a shorthand involving fractions. Consider the problem:

Arrange 51 tiles in 9 rows.

The pupil is to report on his arrangement. There is only *one* possible arrangement:



But there are several possible *reports* of the arrangement. For example, one report might be:

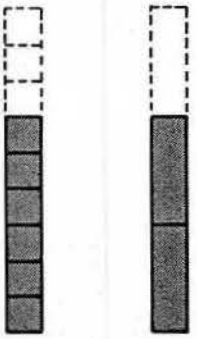
5R6

This report states that the arrangement consists of 5 complete columns and an incomplete column containing 6 tiles. This report does not involve a fraction—it tells you about columns and tiles.

Another report of the arrangement might be:

$$5\frac{6}{9}$$

This report states that the arrangement consists of 5 complete columns and an incomplete column which is $\frac{6}{9}$ full. This report tells you about columns, both complete and incomplete. A column which is $\frac{6}{9}$ full is also $\frac{2}{3}$ full, a fact which can be verified by observation:

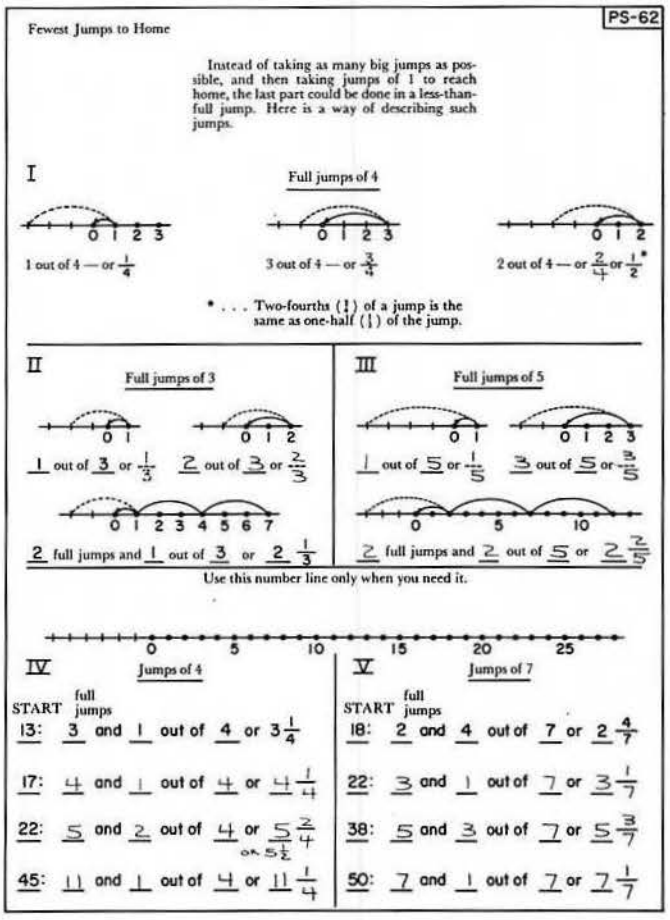


And so we could also report the arrangement by writing:

$$5\frac{2}{3}$$

The report "5R6" gives us different information from either of the reports "$5\frac{6}{9}$" or "$5\frac{2}{3}$". The latter two reports tell us *precisely* the same thing.

Practice sheet 62 introduces a similar method of reporting results for number line jumps.

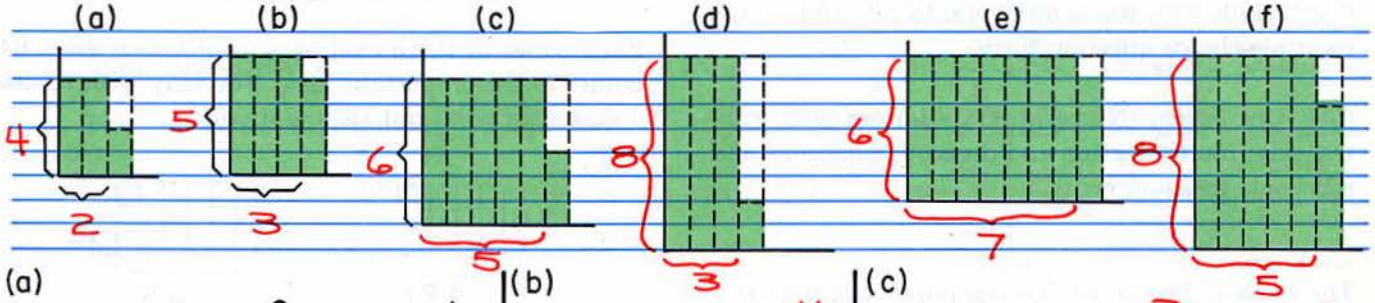

PS-62
Fewest Jumps to Home
Instead of taking as many big jumps as possible, and then taking jumps of 1 to reach home, the last part could be done in a less-than-full jump. Here is a way of describing such jumps.
I
Full jumps of 4
1 out of 4 — or 1/4
3 out of 4 — or 3/4
2 out of 4 — or 2/4 or 1/2*
* . . . Two-fourths (2/4) of a jump is the same as one-half (1/2) of the jump.
II
Full jumps of 3
1 out of 3 or 1/3
2 out of 3 or 2/3
2 full jumps and 1 out of 3 or 2 1/3
III
Full jumps of 5
1 out of 5 or 1/5
3 out of 5 or 3/5
2 full jumps and 2 out of 5 or 2 2/5
Use this number line only when you need it.
0 5 10 15 20 25
IV
Jumps of 4
START full jumps
13: 3 and 1 out of 4 or 3 1/4
17: 4 and 1 out of 4 or 4 1/4
22: 5 and 2 out of 4 or 5 2/4 or 5 1/2
45: 11 and 1 out of 4 or 11 1/4
V
Jumps of 7
START full jumps
18: 2 and 4 out of 7 or 2 4/7
22: 3 and 1 out of 7 or 3 1/7
38: 5 and 3 out of 7 or 5 3/7
50: 7 and 1 out of 7 or 7 1/7

On this page use the more common way of recording results of arrangements of rows and columns and left over squares.

Former way: $3\overline{)14}$ → $4R2$

More common way: $4R2$ over $3\overline{)14}$

Record each of the arrangements sketched below.
Show two or three different ways of recording each.



(a) $4\overline{)10}$ $2R2$, $2\frac{2}{4}$, $2\frac{1}{2}$

(b) $5\overline{)19}$ $3R4$, $3\frac{4}{5}$

(c) $6\overline{)33}$ $5R3$, $5\frac{3}{6}$, $5\frac{1}{2}$

(d) $8\overline{)26}$ $3R2$, $3\frac{2}{8}$, $3\frac{1}{4}$

(e) $6\overline{)47}$ $7R5$, $7\frac{5}{6}$

(f) $8\overline{)46}$ $5R6$, $5\frac{6}{8}$, $5\frac{3}{4}$

Use the grid lines above (if you need to) to draw sketches for any of the records g - o below.

(g) $4\overline{)30}$ $7R2$, $7\frac{2}{4}$, $7\frac{1}{2}$

(h) $5\overline{)44}$ $8R4$, $8\frac{4}{5}$

(i) $6\overline{)20}$ $3R2$, $3\frac{2}{6}$, $3\frac{1}{3}$

(j) $9\overline{)30}$ $3R3$, $3\frac{3}{9}$, $3\frac{1}{3}$

(k) $3\overline{)32}$ $10R2$, $10\frac{2}{3}$

(l) $8\overline{)26}$ $3R2$, $3\frac{2}{8}$, $3\frac{1}{4}$

(m) $10\overline{)45}$ $4R5$, $4\frac{5}{10}$, $4\frac{1}{2}$

(n) $4\overline{)49}$ $12R1$, $12\frac{1}{4}$

(o) $6\overline{)21}$ $3R3$, $3\frac{3}{6}$, $3\frac{1}{2}$

Tasks: In exercises a-f *pupils use conventional notation to record the sketched arrangements. In exercises* g-o *pupils use conventional notation to complete the given records.*

Purpose: To perfect computational skills with division of whole numbers and to provide experience in recording computational results in conventional notation.

Unifying Ideas: Numeration; multiplication and division.

The Lesson: Both methods of reporting are used for all exercises on pupil page 63.

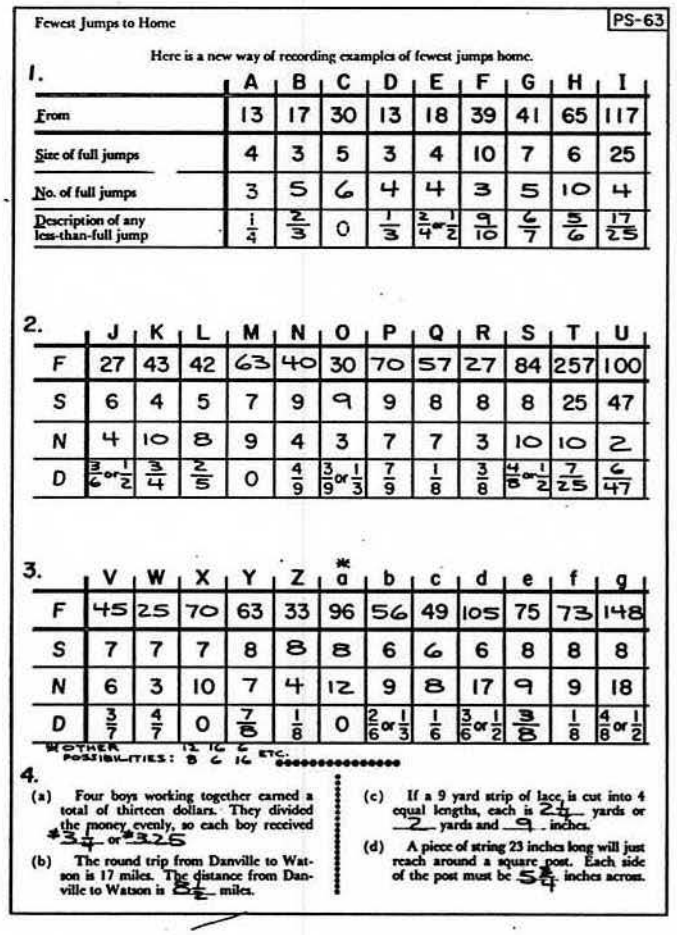
Any objects can be grouped in rectangular arrangements. We can speak of 4 columns of 12 tiles each in much the same way we speak of 4 dozen eggs or 4 feet (since there are 12 inches in a foot) or 4 items that cost 12¢ each or 4 groups of 12 boys each. Further, we notice that 4 columns of 12 tiles have the same number of tiles as 12 columns of 4 tiles. We can see that a dozen groups of 4 eggs is the same as 4 dozen eggs, etc.

How many dozens are in 30? Answer: 2 and 6 left over. In everyday language, this is usually translated into 2½ dozen. How many 60-minute periods (hours) are in 100 minutes? Answer: 1 and $^{4}\%_{0}$ which may be reported as 1⅔ hours or, more probably, 1 hour and 40 minutes.

In all such discussion, we place more emphasis on making understandable reports in everyday language about experiences than on a particular form in our arithmetic. The arithmetic by itself is quite useless—it becomes valuable when it is interpreted as saying something about the world we live in.

When a group of 3 boys is confronted with the task of dividing 10¢ evenly among them, all that arithmetic can do is report that 10 things arranged 3 per group produces 3 groups with 1 left over, or 3⅓ groups. It can help the boys no further with their still unsolved problem.

Deeper insight into this problem will develop as we consider more fully the question of fractions. The discussion of fractions (following the next two checkup pages) begins with developing a means of reporting experiences with counting.



Fewest Jumps to Home — PS-63

Here is a new way of recording examples of fewest jumps home.

1.

	A	B	C	D	E	F	G	H	I
From	13	17	30	13	18	39	41	65	117
Size of full jumps	4	3	5	3	4	10	7	6	25
No. of full jumps	3	5	6	4	4	3	5	10	4
Description of any less-than-full jump	$\frac{1}{4}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{4}$ or $\frac{1}{2}$	$\frac{9}{10}$	$\frac{6}{7}$	$\frac{5}{6}$	$\frac{17}{25}$

2.

	J	K	L	M	N	O	P	Q	R	S	T	U
F	27	43	42	63	40	30	70	57	27	84	257	100
S	6	4	5	7	9	9	9	8	8	8	25	47
N	4	10	8	9	4	3	7	7	3	10	10	2
D	$\frac{3}{6}$ or $\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{5}$	0	$\frac{4}{9}$	$\frac{3}{9}$ or $\frac{1}{3}$	$\frac{7}{9}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{4}{8}$ or $\frac{1}{2}$	$\frac{7}{25}$	$\frac{6}{47}$

3.

	V	W	X	Y	Z	*a	b	c	d	e	f	g
F	45	25	70	63	33	96	56	49	105	75	73	148
S	7	7	7	8	8	8	6	6	6	8	8	8
N	6	3	10	7	4	12	9	8	17	9	9	18
D	$\frac{3}{7}$	$\frac{4}{7}$	0	$\frac{7}{8}$	$\frac{1}{8}$	0	$\frac{2}{6}$ or $\frac{1}{3}$	$\frac{1}{6}$	$\frac{3}{6}$ or $\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{4}{8}$ or $\frac{1}{2}$

ANOTHER POSSIBILITIES: 12 16 6 ETC.

4.

(a) Four boys working together earned a total of thirteen dollars. They divided the money evenly, so each boy received $3\frac{1}{4}$ or $3.25

(b) The round trip from Danville to Watson is 17 miles. The distance from Danville to Watson is $8\frac{1}{2}$ miles.

(c) If a 9 yard strip of lace is cut into 4 equal lengths, each is $2\frac{1}{4}$ yards or 2 yards and 9 inches.

(d) A piece of string 23 inches long will just reach around a square post. Each side of the post must be $5\frac{3}{4}$ inches across.

Following Directions

Some numbers are listed in the blue region. Make a list of pairs of these numbers which fit the following description.

14	3	5	56	23	18
19	12	17	66	2	
4	6	91	8	49	16

1. Their sum is 25 ..... 17, 8 | 23, 2 | 19, 6
2. Their product is 24 ..... 6, 4 | 8, 3 | 12, 2
3. Their difference is 13 ..... 19, 6 | 17, 4 | 16, 3 | 18, 5
4. Their quotient is 4 ..... 16, 4 | 8, 2 | 12, 3 | 56, 14
5. They are both odd numbers ... A FEW POSSIBLE ANS. → 17, 19 | 3, 5 | 91, 49
6. They are both even numbers .. A FEW POSSIBLE ANS. → 2, 8 | 4, 6 | 16, 12
7. Their sum is 10 and their product is 24 ..... 4, 6 | , | ,
8. Their sum is 7 and their difference is 1 ..... 4, 3 | , | ,
9. Their difference is 14 and their product is 51 ..... 17, 3 | , | ,
10. Their product is 36 and their quotient is 4 ..... 12, 3 | , | ,
11. Their difference is 47 and their sum is 85 ..... 66, 19 | , | ,

A nickel is worth

(1) 8¢ (2) 10¢ (3) 5¢ (4) 25¢

Mark correct answer (1) (2) (3) (4)

12 Which number has a 7 in the tens place?

(1) 747 (3) 7,104
(2) 573 (4) 897

(1) (2) (3) (4)

13 How long is it from 10:30 A.M. to 5:30 P.M.?

(1) 7 hrs. (3) 10 hrs.
(2) 5 hrs. (4) 15 hrs.

(1) (2) (3) (4)

14 Which number is 4 ones, 6 tens, and 7 hundreds?

(1) 647 (3) 467
(2) 746 (4) 764

(1) (2) (3) (4)

15 Which number comes next in this sequence? 23, 19, 15.

(1) 14 (3) 11
(2) 10 (4) 16

(1) (2) (3) (4)

16 What is the cost of a dozen pens if 3 pens cost 35¢?

(1) 95¢ (3) $1.40
(2) 70¢ (4) $1.15

(1) (2) (3) (4)

17 If 2 erasers cost 16¢, how many can you buy for 48¢?

(1) 6 (3) 8
(2) 3 (4) 24

(1) (2) (3) (4)

18 17 x 3 =

(1) 51 (3) 37
(2) 73 (4) 42

(1) (2) (3) (4)

19 2 is $\frac{1}{3}$ of what number?

(1) 23 (3) 7
(2) 222 (4) 6

(1) (2) (3) (4)

20 What number is XXXVI?

(1) 51 (3) 53
(2) 36 (4) 34

(1) (2) (3) (4)

Tasks: *In exercises 1-11 pupils find pairs of numbers in the blue region that fit the given descriptions. In exercises 12-20 pupils mark the correct answers to the questions as illustrated in the example.*

Purpose: To assess knowledge of a variety of mathematical concepts and achievement of computational skills with whole numbers; to reintroduce the multiple-choice question form.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Before letting the pupils begin, pose a few problems to be sure they are familiar with the words used—*sum, product, difference, quotient, odd,* and *even.*

I'm thinking of a pair of numbers; their sum is 10; their difference is 2. What are the numbers? (4 and 6)

In the statement "$36 \div 9 = 4$" is the quotient an odd or an even number? (Even) Is the divisor odd or even? (Odd) Is the quotient a multiple of 3? (No)

What is the product of the two largest odd numbers less than 10? (63)

Further, be sure that pupils understand that questions may have several answers:

The sum of two whole numbers is 5. What might the numbers be? (0 and 5, 1 and 4, 2 and 3)

The product of a pair of whole numbers is 30. List all such pairs. (1 and 30, 2 and 15, 3 and 10, 5 and 6)

Practice sheet 64 is an extension of the preceding four pupil pages and practice sheets. The examples are carefully planned to help pupils develop a deeper understanding of the process of division. They may realize that there is a division version of the distributive principle. Here is its pattern:

$$(a+b) \div c = (a \div c) + (b \div c)$$

They should recognize it as the idea behind the regrouping in this example:

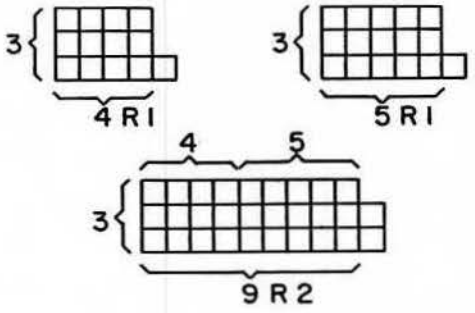
$$7\overline{)105} = 7\overline{)70} + 7\overline{)35} = 10 + 5 = 15$$

They may be surprised by the following result:

$$7\overline{)23}\ 3\text{R}2 \qquad 7\overline{)31}\ 4\text{R}3 \qquad 7\overline{)54}\ 7\text{R}5$$

$23 + 31 = 54$ (dividends)
$3 + 4 = 7$ (partial quotients)
$2 + 3 = 5$ (remainders)

Each series of three examples on practice sheet 64 points to this important idea. You may wish to use a sketch to point out this relationship:



PS-64

$3\overline{)23}^{7\text{R}2}$ represents many things in the world. Here are two of them. What other things can you think of?

Number line Jumps

full jumps of size shown
size of a full jump → $3\overline{)23}^{7\text{R}2}$ ← 1 unit jumps
starting point on a number line

Column and Row Arrangements

number of full columns
size of a full column → $3\overline{)23}^{7\text{R}2}$ ← left over in last column
number of tiles

a.	$4\overline{)29}$ 7R1	$4\overline{)42}$ 10R2	$4\overline{)71}$ 17R3	b.	$8\overline{)50}$ 6R2	$8\overline{)65}$ 8R1	$8\overline{)115}$ 14R3
c.	$5\overline{)62}$ 12R2	$5\overline{)47}$ 9R2	$5\overline{)109}$ 21R4	d.	$7\overline{)40}$ 5R5	$7\overline{)51}$ 7R2	$7\overline{)91}$ 13R0
e.	$10\overline{)103}$ 10R3	$10\overline{)71}$ 7R1	$10\overline{)174}$ 17R4	f.	$9\overline{)60}$ 6R6	$9\overline{)40}$ 4R4	$9\overline{)100}$ 11R1
g.	$6\overline{)68}$ 11R2	$6\overline{)32}$ 5R2	$6\overline{)100}$ 16R4	h.	$12\overline{)75}$ 6R3	$12\overline{)50}$ 4R2	$12\overline{)125}$ 10R5
i.	$15\overline{)150}$ 10R0	$15\overline{)50}$ 3R5	$15\overline{)200}$ 13R5	j.	$25\overline{)507}$ 20R7	$25\overline{)225}$ 9R0	$25\overline{)732}$ 29R7

Problems

1. If Mr. Jones divided a group of stamps evenly among his 6 children, he would have 4 left over. If they had more than 25 and less than 35 stamps to start with, then they must have had either 28 or 34 stamps. If there were 2 more stamps to begin with, then each child would have had either 5 or 6 when Mr. Jones divided them evenly among the 6 children.

2. The largest number of stamps less than 50 that 8 people can divide evenly among themselves is 48.

3. There is a number between 25 and 50 that, when divided by 5 has a remainder of 2 and when divided by 4 has a remainder of 1. The number must be 37.

In each of the Exercises I-IV, indicate the largest number by writing an A in the blue block, the next largest by a B, the next largest by a C, the smallest by a D, and then compute the differences asked for.

I

Seventy	70	C
Seven thousand	7000	A
Seven hundred	700	B
Seven	7	D

difference between

A and B 6,300 B and D 693

B and C 630 C and D 63

A and C 6,930 A and D 6,993

II

427	Four hundred twenty-seven	C
742	Seven hundred forty-two	A
274	Two hundred seventy-four	D
472	Four hundred seventy-two	B

difference between

A and B 270 B and D 198

B and C 45 C and D 153

A and C 315 A and D 468

III

7×9	63	C
$35 + 47$	82	B
$100 \div 2$	50	D
$207 - 109$	98	A

difference between

A and B 16 B and D 32

B and C 19 C and D 13

A and C 35 A and D 48

IV

$6.50	13 half-dollars	C
$7.70	6 dollars and 17 dimes	A
$5.25	4 dollars and 25 nickels	D
$7.50	3 dollars and 18 quarters	B

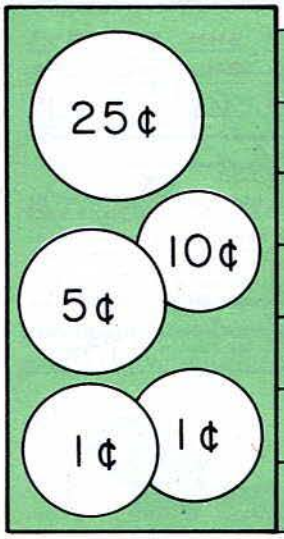
difference between

A and B $.20 B and D $2.25

B and C $1.00 C and D $1.25

A and C $1.20 A and D $2.45

V



amount shown	42 ¢	seventeen cents less	25 ¢
twice as much	84 ¢	thirty-nine cents more	81 ¢
half as much	21 ¢	needed to make 60¢	18 ¢
three times as much	$1.26	needed to make $2.00	$1.58
one third as much	14 ¢	eighty-five cents more	$1.27
four times as much	$1.68	number of coins	5
ten times as much	$4.20	10 times as many coins	50

Notes on Pupil Page 65

Tasks: In exercises I-IV pupils label the given numbers in order of size and compute the required differences. In exercise V they determine the number represented by the picture and carry out the indicated computations.

Purpose: To assess pupils' knowledge of a variety of mathematical concepts and computational skills with whole numbers.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; measurement.

The Lesson: Pupil page 65, like pupil page 64, may be used as a checkup. You may wish to familiarize the pupils with the kinds of questions asked. For exercises I-IV have them arrange the numbers 124, 412, 214, and 241 in order:

124	214	241	412
A	B	C	D

What are the differences between:

A and B: 90 B and D: 198

B and C: 27 C and D: 171

A and C: 117 A and D: 288

For exercise V sketch coins on the chalkboard:

How much is shown? How many coins are shown? What is twice as much? Half as much? A third as much? How much more is needed to make 50¢? $1.00? 75¢? How many coins would there be if there were twice as many? Three times as many?

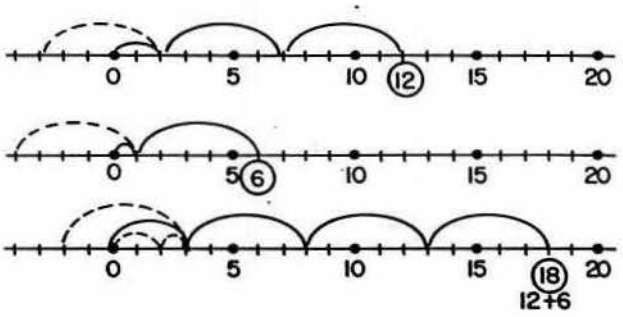
Practice sheet 65 expands the ideas on practice sheet 64 and the four preceding pupil pages and practice sheets. Again, as on practice sheet 64, the examples are chosen and arranged to shed more light on division with remainders. The results may help anticipate addition of fractions, although we do not expect any extensive discussion of this now.

$7\overline{)31}^{\,4\frac{3}{7}}$, $7\overline{)44}^{\,6\frac{2}{7}}$, $7\overline{)75}^{\,10\frac{5}{7}}$

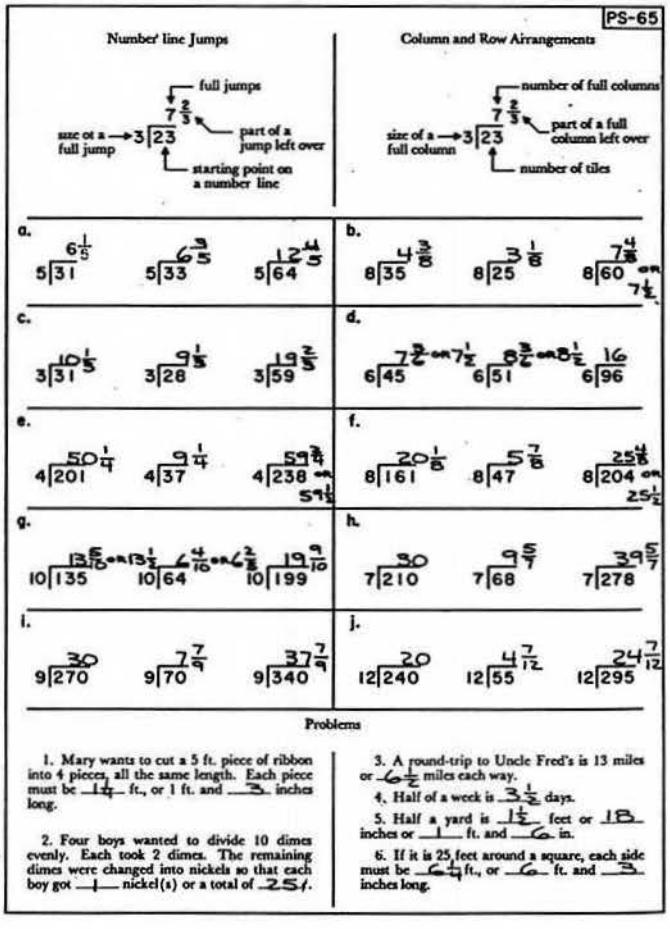
Pupils may notice, after several examples, that:

$31 + 44 = 75$ (dividends)
$4 + 6 = 10$ (partial quotients)
$3 + 2 = 5$ (remainders)

Again, you may wish to draw a sketch:



$5\overline{)12}^{\,2\frac{2}{5}}$, $5\overline{)6}^{\,1\frac{1}{5}}$, $5\overline{)18}^{\,3\frac{3}{5}}$


PS-65
Number line Jumps
Column and Row Arrangements
full jumps
size of a full jump
part of a jump left over
starting point on a number line
number of full columns
size of a full column
part of a full column left over
number of tiles
a.
b.
c.
d.
e.
f.
g.
h.
i.
j.
Problems
1. Mary wants to cut a 5 ft. piece of ribbon into 4 pieces, all the same length. Each piece must be 1¼ ft., or 1 ft. and 3 inches long.
2. Four boys wanted to divide 10 dimes evenly. Each took 2 dimes. The remaining dimes were changed into nickels so that each boy got 1 nickel(s) or a total of 25¢.
3. A round-trip to Uncle Fred's is 13 miles or 6½ miles each way.
4. Half of a week is 3½ days.
5. Half a yard is 1½ feet or 18 inches or 1 ft. and 6 in.
6. If it is 25 feet around a square, each side must be 6¼ ft., or 6 ft. and 3 inches long.

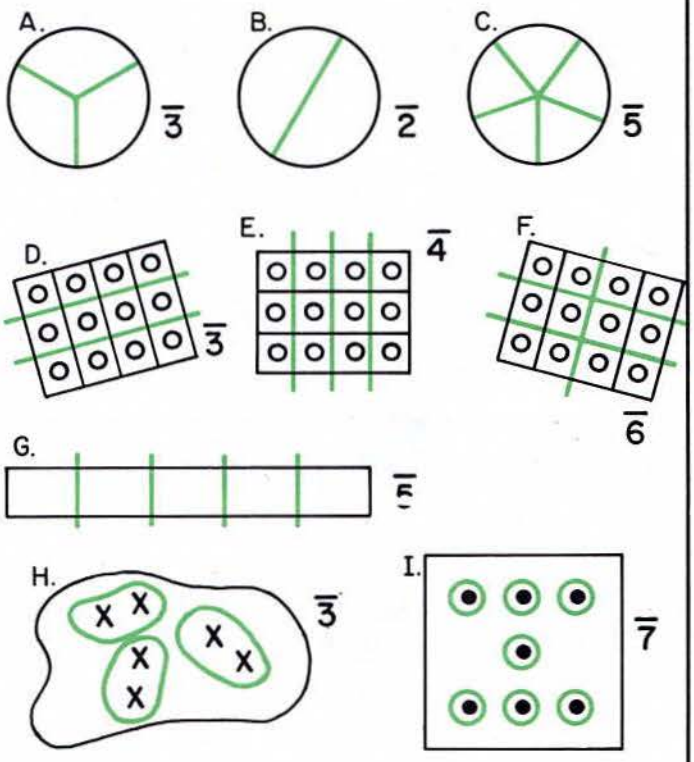
Ben asked Mrs. Brown whether she knew what he meant by this:

$\overline{3}, 2$

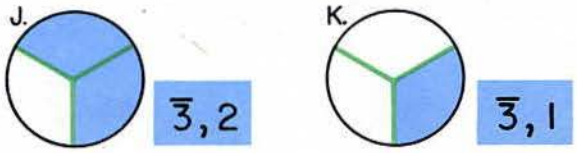
"No, I don't," Mrs. Brown was quick to admit. "What do you mean by that, Ben?"

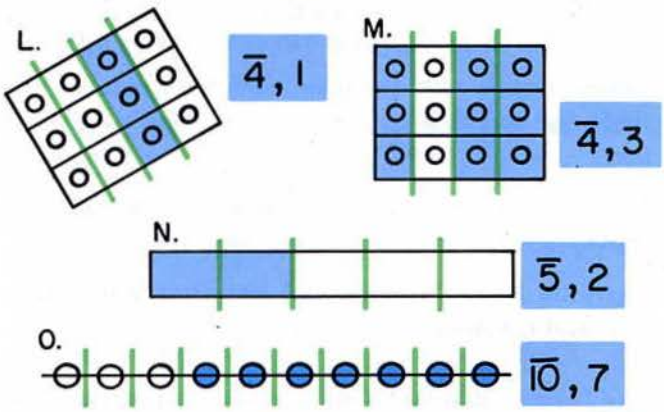
"I mean: cut (or separate) whatever we are talking about into 3 parts of equal size and color 2 of them — or paint out any 2 of them.

"First, I'll explain $\overline{3}$. I call that 'Bar three.' I call $\overline{2}$, 'Bar two'; $\overline{4}$, 'Bar four'; $\overline{7}$, 'Bar seven'; $\overline{12}$, 'Bar twelve'; etc.

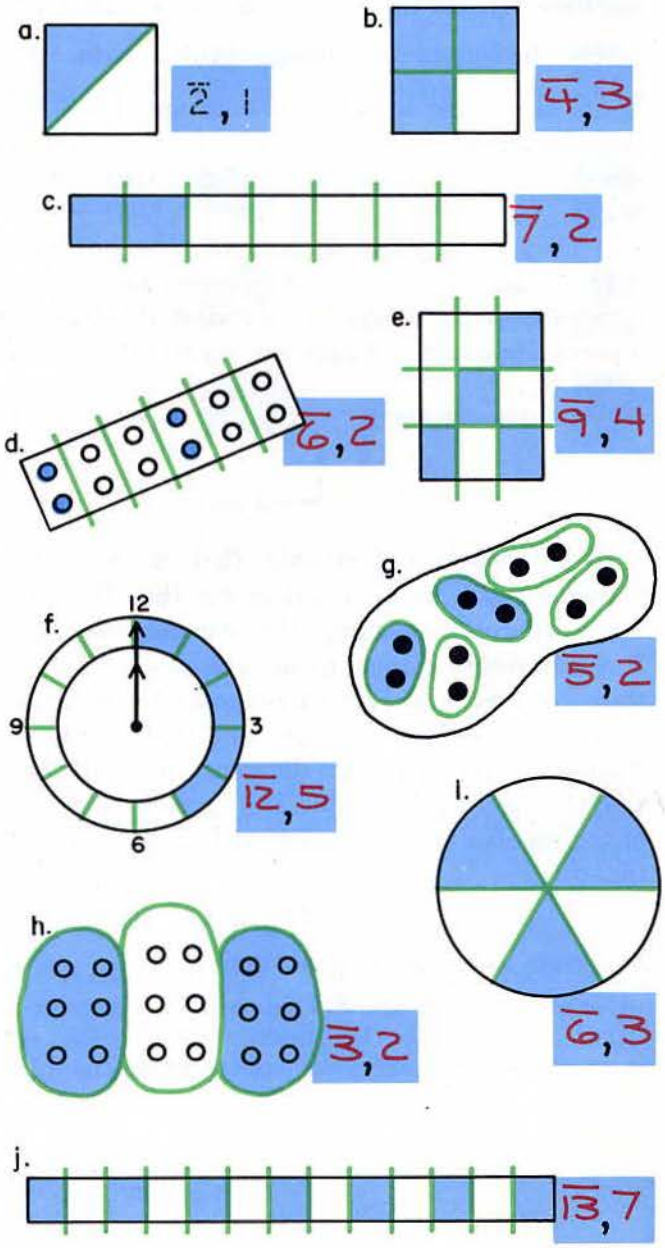


"The second number I write tells you to color a certain number of the parts I've cut or separated the original thing into.





"Now I will make some sketches and you can write in the blue blocks the directions for cutting and coloring."



Tasks: In exercises A-I pupils study the pictures and notation used to describe the method of partitioning each object or collection. In exercises J-O they study the notation that describes both the partitioning and coloring for each picture. In exercises a-j *pupils use the new notation that describes both the partitioning and coloring for each picture.*

Purpose: To introduce concepts of fraction and a notation for fractions.

Unifying Ideas: Number and counting; multiplication and division.

The Lesson: This section is designed to help pupils understand how we use fractions in talking about events in the world around us. Fractions arise out of problems we face in measurement. Counting numbers are enough when we talk about whole numbers of apples or cups or stars. They are enough as long as we have chosen a unit of measurement such as pounds, and are satisfied to report that the weight of something is more than 6 pounds and less than 7 pounds.

When we wish to give more information than "more than 11 feet and less than 12 feet," or "more than 0 and less than 1 pie," we need to extend our concept of number beyond that of counting numbers.

We decide that the unit we have been working with is too large to give an adequate description. So we select a smaller unit; that is, we change the "denomination." We break our unit—such as feet—up into a number of parts that are the same size, and then report our measurement as a certain number of this new denomination. We indicate the new unit or new denomination by what we call the denominator and we record the number of these new units by what we call the numerator.

numerator	(number of)
denominator	(new units)

If we are reporting a measurement of weeks and are not satisfied with "less than 1 week," we divide the week into seven parts—indicated by:

$\overline{7}$ (a week divided into 7 parts)

and then record the time as, perhaps, 3 of these new units. Our completed report:

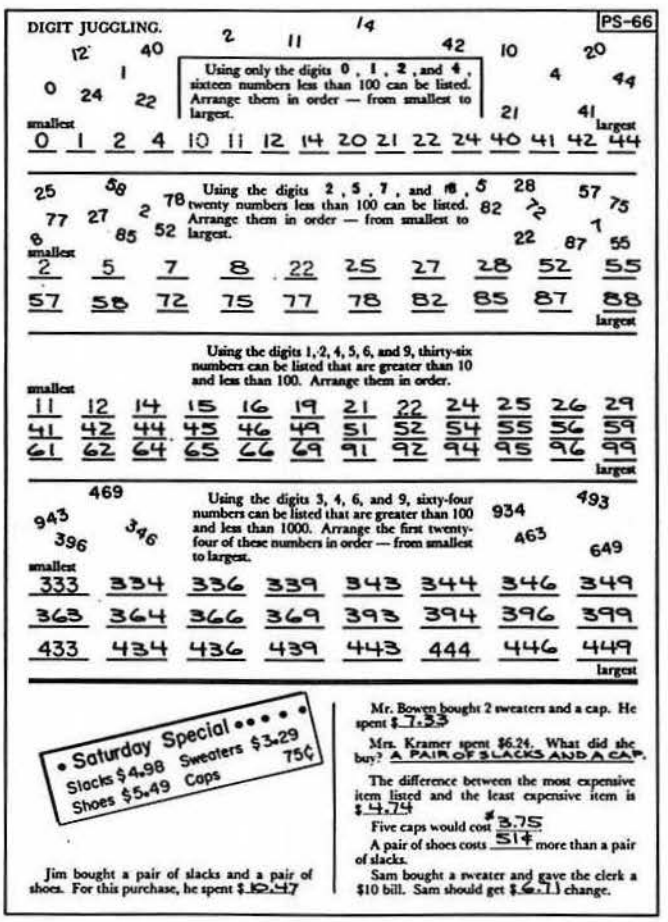
$\frac{3}{7}$ weeks (3 of the parts of a week)
(a week divided into 7 parts)

So a fraction reports two very different things:

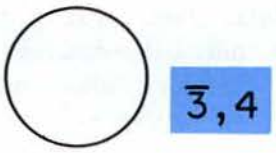
(a) it reports the new unit or denomination we are using;
(b) it reports our count of these new units.

Ben's novel way of reporting these things helps focus on the idea we are developing.

Practice sheet 66 introduces a new game called *Digit Juggling*. As this game unfolds, you will find it requires extensive practice in computation and contributes much to a real understanding of place-value.


DIGIT JUGGLING.
PS-66
Using only the digits 0, 1, 2, and 4, sixteen numbers less than 100 can be listed. Arrange them in order — from smallest to largest.
smallest
0 1 2 4 10 11 12 14 20 21 22 24 40 41 42 44
largest
Using the digits 2, 5, 7, and 8, twenty numbers less than 100 can be listed. Arrange them in order — from smallest to largest.
smallest
2 5 7 8 22 25 27 28 52 55
57 58 72 75 77 78 82 85 87 88
largest
Using the digits 1, 2, 4, 5, 6, and 9, thirty-six numbers can be listed that are greater than 10 and less than 100. Arrange them in order.
smallest
11 12 14 15 16 19 21 22 24 25 26 29
41 42 44 45 46 49 51 52 54 55 56 59
61 62 64 65 66 69 91 92 94 95 96 99
largest
Using the digits 3, 4, 6, and 9, sixty-four numbers can be listed that are greater than 100 and less than 1000. Arrange the first twenty-four of these numbers in order — from smallest to largest.
smallest
333 334 336 339 343 344 346 349
363 364 366 369 393 394 396 399
433 434 436 439 443 444 446 449
largest
Saturday Special
Slacks $4.98 Sweaters $3.29
Shoes $5.49 Caps 75¢
Jim bought a pair of slacks and a pair of shoes. For this purchase, he spent $10.47
Mr. Bowen bought 2 sweaters and a cap. He spent $7.33
Mrs. Kramer spent $6.24. What did she buy? A PAIR OF SLACKS AND A CAP.
The difference between the most expensive item listed and the least expensive item is $4.74
Five caps would cost $3.75
A pair of shoes costs $5.14 more than a pair of slacks.
Sam bought a sweater and gave the clerk a $10 bill. Sam should get $6.71 change.

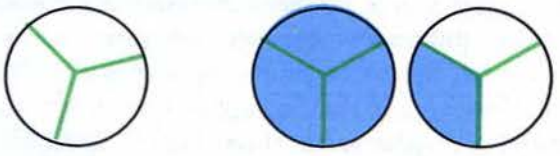
"It's very important," Ben told Mrs. Brown and the class, "to know what it is you're talking about. Suppose I gave you the following directions:



$\overline{3}, 4$

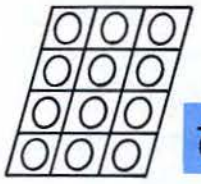
The $\overline{3}$ is easy enough. But how can you color in 4 parts if you only have 3 of them?

"Since I'm talking about a circle of the size shown, you will need to consider more than one of them.



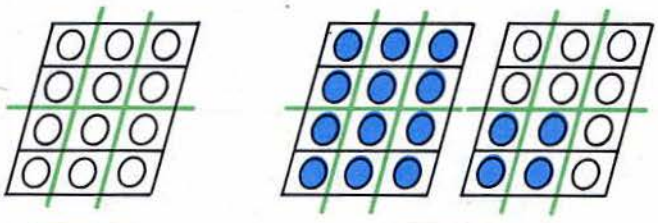
$\overline{3},$ and $\overline{3}, 4$

"Or suppose I was talking about a dozen eggs, and wrote:



$\overline{6}, 8$

"Again, I can do the $\overline{6}$, but I will need more dozens of eggs to show 8 such parts of a dozen."



$\overline{6},$ and $\overline{6}, 8$

At this point, Ruth interrupted.

"Really, Ben, I think you ought to tell everyone the simple way we write those directions."

"Don't tell us," Barbara said, "I think I've guessed."

"Let's hear your guess," Ben answered.

Barbara explained:

If you write $\overline{3}, 2$ .... I would write $\frac{2}{3}$

If you write $\overline{2}, 1$ .... I would write $\frac{1}{2}$

If you write $\overline{7}, 4$ .... I would write $\frac{4}{7}$

If you write $\overline{10}, 7$ .... I would write $\frac{7}{10}$

If you write $\overline{3}, 4$ .... I would write $\frac{4}{3}$

If you write $\overline{6}, 8$ .... I would write $\frac{8}{6}$

"Those are fractions," Harry exclaimed with surprise.

"Ruth always gives things away," Ben complained.

"Not really this time," Mrs. Brown suggested, "since Harry and some others in the class began to recognize your idea as a familiar one."

"Well," Ben said, "I've got some 'Iso Cards' I invented. They also have to do with the idea we are talking about. They are all about Things and Numbers."

Ben prepared the next several pages. Can you discover his rules?

Notes on Pupil Page 67

Tasks: *Pupils read the story used to introduce fractions and complete the record in Barbara's explanation.*

Purpose: To develop a rationale for fraction concepts.

Unifying Ideas: Number and counting; multiplication and division.

The Lesson: Ben starts with a unit—a pie, perhaps. The first part of his report tells how he got a new unit:

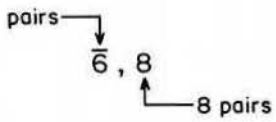
$\overline{3}$ . . . dividing the unit into 3 parts of the same size

He then reports the number of these new units:

4 . . . he counts the number of these parts.

There is nothing at all surprising about the fact that the number in the second part of his report is larger than the number in the first part of his report.

If he isn't satisfied to note that he has more than 1 dozen and less than 2 dozen, he selects a new unit—pairs of eggs—by dividing the dozen in 6 parts. He counts his eggs and reports 8 pairs:



It is perhaps unfortunate that we read the numerator of a fraction first rather than following Ben's approach—reading the denominator first. It would sound clumsy to us now to say "fourths three" and we make no such proposal. However, we do feel it is extremely important that pupils see fractions as reporting two different kinds of activities: (1) selecting a new unit, and (2) making a count of the number of those units. After Ben's explanation of:

$\overline{3}$, 2

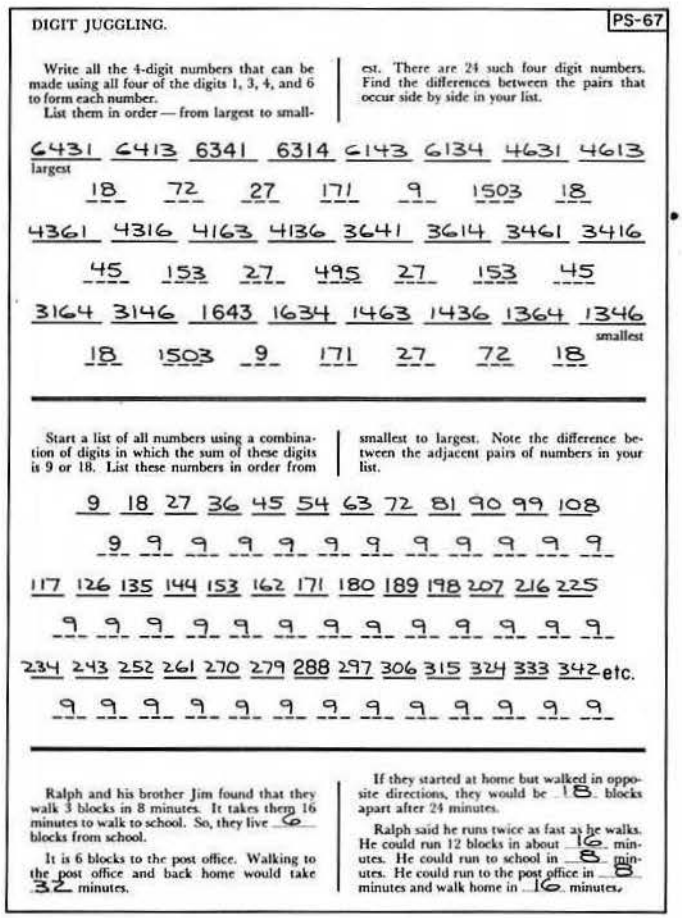
pupils are more apt to understand that:

$\frac{2}{3}$

is simply our customary way of reporting the same activity.

However, since pupils are already familiar with simple fractions, there is no point in maintaining any disguise—so Ruth and Barbara spoil Ben's game. And, while we will take a look at Ben's "Iso Cards," you can rest assured he will soon drop his unusual notation—and familiar fractions will be used throughout.

In *Digit Juggling*, as extended on practice sheet 67, the number 9 begins to recur frequently. In the first section, all the small differences are easily recognizable as multiples of 9. Some pupils may wonder about 171, 1503, 153 and 495. If they carry out the necessary computations, they will find that these, too, are multiples of 9. Nine appears in the second activity in a more dramatic way. This recurrence of 9 is a characteristic of our decimal system of numeration, as *Digit Juggling* will help reveal. (If you can't wait, you can read Notes on Pupil Page 70 to find out why this happens.)



DIGIT JUGGLING. PS-67

Write all the 4-digit numbers that can be made using all four of the digits 1, 3, 4, and 6 to form each number.
List them in order — from largest to smallest. There are 24 such four digit numbers. Find the differences between the pairs that occur side by side in your list.

6431 6413 6341 6314 6143 6134 4631 4613 (largest)
18 72 27 171 9 1503 18
4361 4316 4163 4136 3641 3614 3461 3416
45 153 27 495 27 153 45
3164 3146 1643 1634 1463 1436 1364 1346 (smallest)
18 1503 9 171 27 72 18

Start a list of all numbers using a combination of digits in which the sum of these digits is 9 or 18. List these numbers in order from smallest to largest. Note the difference between the adjacent pairs of numbers in your list.

9 18 27 36 45 54 63 72 81 90 99 108
9 9 9 9 9 9 9 9 9 9 9 9
117 126 135 144 153 162 171 180 189 198 207 216 225
9 9 9 9 9 9 9 9 9 9 9 9 9
234 243 252 261 270 279 288 297 306 315 324 333 342 etc.
9 9 9 9 9 9 9 9 9 9 9 9 9

Ralph and his brother Jim found that they walk 3 blocks in 8 minutes. It takes them 16 minutes to walk to school. So, they live 6 blocks from school.

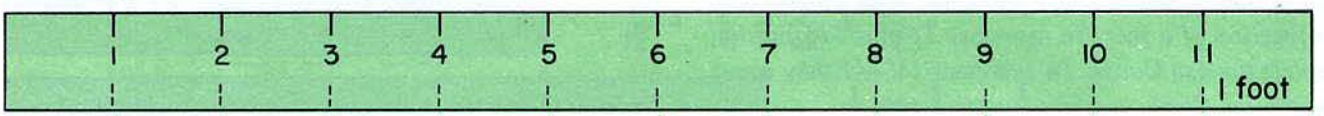
It is 6 blocks to the post office. Walking to the post office and back home would take 32 minutes.

If they started at home but walked in opposite directions, they would be 18 blocks apart after 24 minutes.

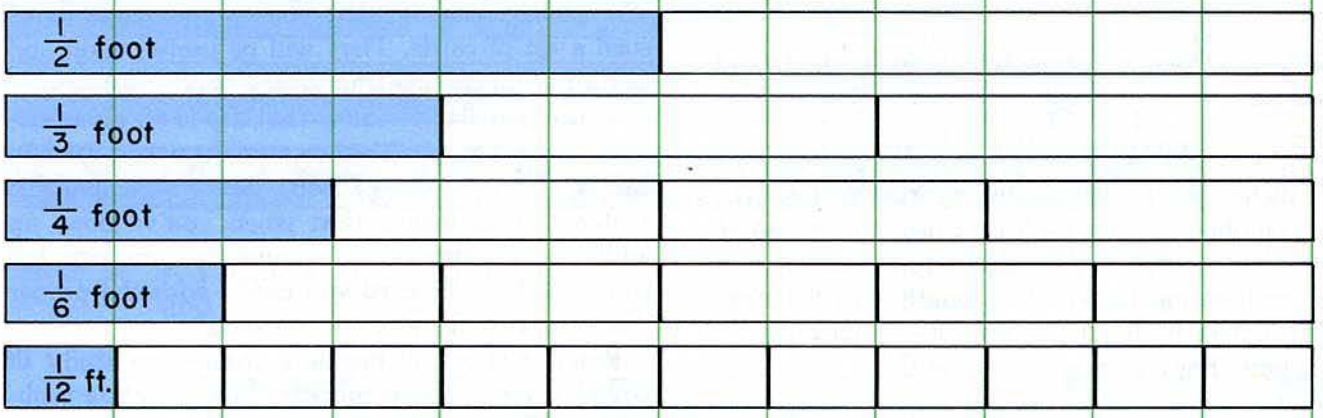
Ralph said he runs twice as fast as he walks. He could run 12 blocks in about 16 minutes. He could run to school in 8 minutes. He could run to the post office in 8 minutes and walk home in 16 minutes.

Ben's Report

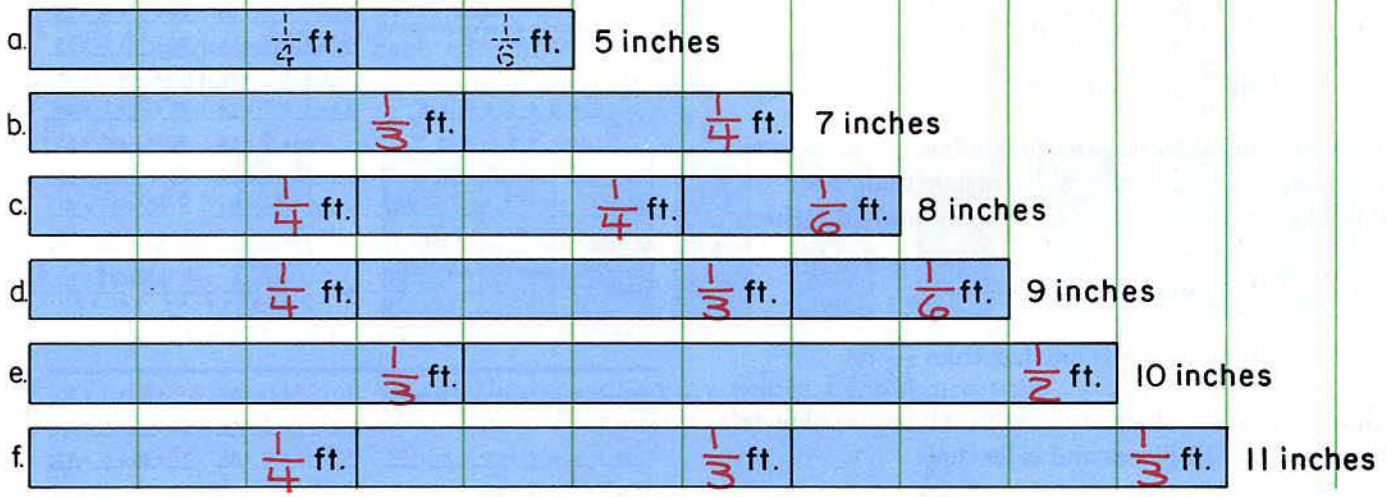
I am thinking about a 1 foot ruler.
Using the inch marks only, how many different ways can I divide the 1 foot ruler up into pieces of the same length? How can I talk about each length (in inches) as a part of a foot?



I find I can (using the inch markings) divide a foot stick in 2 parts, 3 parts, 4 parts, 6 parts, and 12 parts.



I can use combinations of these sticks to make sticks 5, 7, 8, 9, 10, and 11 inches long. Here are my first combinations. You can label the sticks I used.



Let's label these sticks to show the way they are made up:

A. 5 in.: $\frac{1}{4}$ ft. + $\frac{1}{6}$ ft.

B. 7 in.: $\frac{1}{3}$ ft. + $\frac{1}{4}$ ft.

C. 8 in.: $\frac{1}{4}$ ft. + $\frac{1}{4}$ ft. + $\frac{1}{6}$ ft.

D. 9 in.: $\frac{1}{4}$ ft. + $\frac{1}{3}$ ft. + $\frac{1}{6}$ ft.

E. 10 in.: $\frac{1}{3}$ ft. + $\frac{1}{2}$ ft.

F. 11 in.: $\frac{1}{4}$ ft. + $\frac{1}{3}$ ft. + $\frac{1}{3}$ ft.

Tasks: In exercises a-f *pupils view each blue bar as a combination of two or more shorter bars and label each shorter bar as a fraction of a foot. In exercises A-F pupils make a record of their work in exercises* a-f.

Purpose: To introduce the addition of fractions.

Unifying Ideas: Number and counting; addition and subtraction; measurement.

The Lesson: When pupils first hear of one-half, one-third, one-fourth, and three-fourths, it is always in a context in which they understand the expression, as: "one-half of something," "three-fourths of something," etc.

We shall continue to discuss fractions in this way (as a branch of the *Arithmetic of Things and Numbers*) for some time, until pupils are ready to consider fractions as abstract entities. We shall continue to report each new unit and the number of these new units.

Our first exploration is concerned with a foot as the unit. Ben begins by trying to find out in how many ways he can divide a ruler into parts of equal length, using only the inch markings. He finds he can divide it into 2 parts, 3 parts, 4 parts, 6 parts, and 12 parts. He might have reported this activity as:

$$\overline{2},\ \overline{3},\ \overline{4},\ \overline{6},\ \overline{12}$$

But he is using usual notation, so he labels each resulting sub-unit:

$\frac{1}{12}$ foot $\quad \frac{1}{4}$ foot $\quad \frac{1}{2}$ foot

$\frac{1}{6}$ foot $\quad \frac{1}{3}$ foot

to show that he is talking about 1 of each. He can now talk about 1, 2, 3, 4, and 6 inches in terms of a foot—his basic unit.

How can he talk about 5, 7, 8, 9, 10, or 11 inches in terms of feet? He notices that each can be made up of a combination of his new units. You might like to explore other combinations he could have used, such as:

5 inches: $\frac{1}{12}$ ft $+ \frac{1}{3}$ ft or $\frac{1}{6}$ ft $+ \frac{1}{6}$ ft $+ \frac{1}{12}$ ft, etc.

7 inches: $\frac{1}{2}$ ft $+ \frac{1}{12}$ ft or $\frac{1}{4}$ ft $+ \frac{1}{4}$ ft $+ \frac{1}{12}$ ft, etc.

8 inches: $\frac{1}{2}$ ft $+ \frac{1}{6}$ ft or $\frac{1}{6}$ ft $+ \frac{1}{6}$ ft $+ \frac{1}{6}$ ft $+ \frac{1}{6}$ ft, etc.

You may wish to cut out strips 1, 2, 3, 4, and 6 inches long and label them $\frac{1}{12}$ ft, $\frac{1}{6}$ ft, etc. Then the combinations that make different lengths can be demonstrated. You might also consider showing 18 inches:

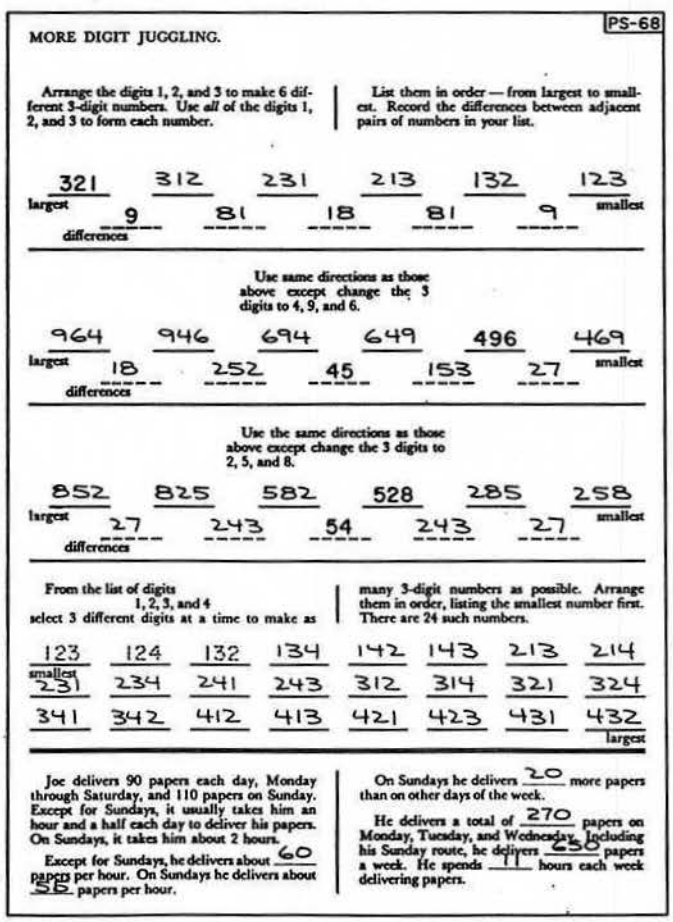
18 inches $= \frac{1}{2}$ ft $+ \frac{1}{2}$ ft $+ \frac{1}{2}$ ft

or $\frac{1}{2}$ ft $+ \frac{1}{3}$ ft $+ \frac{1}{4}$ ft $+ \frac{1}{6}$ ft $+ \frac{1}{6}$ ft $+ \frac{1}{12}$ ft

and include 1 ft $+ \frac{1}{2}$ ft., usually written $1\frac{1}{2}$ ft.

All such activity is good preparation for addition of fractions.

Practice sheet 68 extends *Digit Juggling*, and again multiples of 9 persistently recur. Pupils who expect the differences between adjacent numbers in the fourth list to be multiples of 9 will be surprised. However, if they compare only those that are made with the same three digits—such as 1, 2, and 3—multiples of 9 reappear!



PS-68

MORE DIGIT JUGGLING.

Arrange the digits 1, 2, and 3 to make 6 different 3-digit numbers. Use *all* of the digits 1, 2, and 3 to form each number.

List them in order — from largest to smallest. Record the differences between adjacent pairs of numbers in your list.

largest					smallest
321	312	231	213	132	123
differences: 9	81	18	81	9	

Use same directions as those above except change the 3 digits to 4, 9, and 6.

largest					smallest
964	946	694	649	496	469
differences: 18	252	45	153	27	

Use the same directions as those above except change the 3 digits to 2, 5, and 8.

largest					smallest
852	825	582	528	285	258
differences: 27	243	54	243	27	

From the list of digits 1, 2, 3, and 4 select 3 different digits at a time to make as many 3-digit numbers as possible. Arrange them in order, listing the smallest number first. There are 24 such numbers.

smallest							
123	124	132	134	142	143	213	214
231	234	241	243	312	314	321	324
341	342	412	413	421	423	431	432
							largest

Joe delivers 90 papers each day, Monday through Saturday, and 110 papers on Sunday. Except for Sundays, it usually takes him an hour and a half each day to deliver his papers. On Sundays, it takes him about 2 hours.

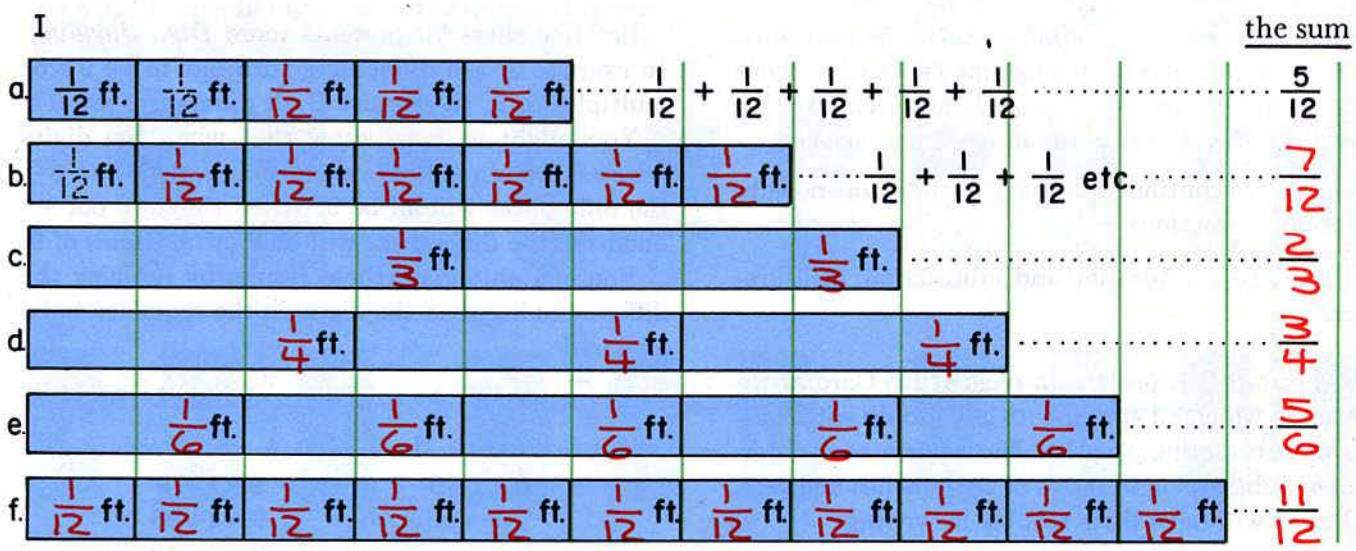
Except for Sundays, he delivers about 60 papers per hour. On Sundays he delivers about 55 papers per hour.

On Sundays he delivers 20 more papers than on other days of the week.

He delivers a total of 270 papers on Monday, Tuesday, and Wednesday. Including his Sunday route, he delivers 590 papers a week. He spends 11 hours each week delivering papers.

The labels I used were clumsy labels. So, I made each of the lengths (5, 7, 8, 9, 10, and 11 inches) out of the fewest pieces of equal length I could. I also made up new labels that told me which piece, and the number of such pieces I used.

I



I made a set of what I call Iso Cards, listing a number of inches on one side, and wrote the new label on the other side. Please fill some of them in.

II

	a	b	c	d	e	f	g	h	i	j	k	l
inches	1	2	3	4	5	6	7	8	9	10	11	12
feet	1/12	1/6	1/4	1/3	5/12	1/2	7/12	2/3	3/4	5/6	11/12	1

I made several of each card and expanded the deck by thinking about lengths that were longer than 12 inches. My plan for extending the deck will be obvious. Please help me.

III

	a	b	c	d	e	f	g	h	i	j
inches	0	13	14	15	16	20	24	29	33	46
feet	0	$1\frac{1}{12}$	$1\frac{1}{6}$	$1\frac{1}{4}$	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{5}{12}$	$2\frac{3}{4}$	$3\frac{5}{6}$

I use these cards to say things I know are true about inches. Please fill in the cards I've left blank.

IV

6	+	3	=	9
1/2	+	1/4	=	3/4

V

8	−	3	=	5
2/3	−	1/4	=	5/12

VI

6	+	3	+	3	=	12
1/2	+	1/4	+	1/4	=	1

Now, it made sense for me to talk either about inches or about feet. I know that 6 inches plus 3 inches is 9 inches.

Then I can say:

$$\frac{1}{2}\text{ ft.} + \frac{1}{4}\text{ ft.} = \frac{3}{4}\text{ ft.}$$

This is only the first set of cards of this type I made.

Tasks: *In exercise I pupils view each blue bar as a combination of shorter bars and label each shorter bar as a fraction of a foot. In exercises II and III pupils complete the Iso Cards. In exercises IV-VI they complete the Iso Cards to make true sentences.*

Purpose: To study the addition and subtraction of fractions.

Unifying Ideas: Addition and subtraction; measurement.

The Lesson: Ben is not satisfied with such clumsy labels as:

$$\frac{1}{4}\text{ ft} + \frac{1}{3}\text{ ft} + \frac{1}{6}\text{ ft}$$

for 9 inches, so he tries another approach. He works out a combination for each that uses the fewest of his original pieces—and insists that all pieces in any one combination be equal in length. For 5 inches and 7 inches, he has no alternative; each must be $\frac{1}{12}$ ft long. For 8 inches, he notices that

$$8\text{ inches} = \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft} + \frac{1}{12}\text{ ft}$$

$$\text{or } \frac{1}{6}\text{ ft} + \frac{1}{6}\text{ ft} + \frac{1}{6}\text{ ft} + \frac{1}{6}\text{ ft}$$

$$\text{or } \frac{1}{3}\text{ ft} + \frac{1}{3}\text{ ft}$$

Since he wants to use the fewest pieces, he selects the last, which he calls $\frac{3}{4}$ ft (rather than $\frac{6}{8}$ or $\frac{9}{12}$ ft). The same happens with 9 inches and 10 inches:

$$9\text{ inches} = \frac{3}{4}\text{ ft (rather than } \frac{9}{12}\text{ ft)}$$

$$10\text{ inches} = \frac{5}{6}\text{ ft (rather than } \frac{10}{12}\text{ ft)}$$

But he has no alternative with 11 inches. He is forced to use 11 pieces and calls that:

$$11\text{ inches} = \frac{11}{12}\text{ ft}$$

This brings Ben to his "Iso Cards." He writes a report of a number of inches on one side of the card and on the other side a report of the same length in feet. Then he expands his set of cards so he can talk about measurements greater than 1 foot.

Now he is ready to play *Iso Card Arithmetic* with his reports of lengths. He can use his cards to say something about adding numbers of inches. For example, 6 inches plus 3 inches equals 9 inches:

$$\boxed{6} + \boxed{3} = \boxed{9}$$

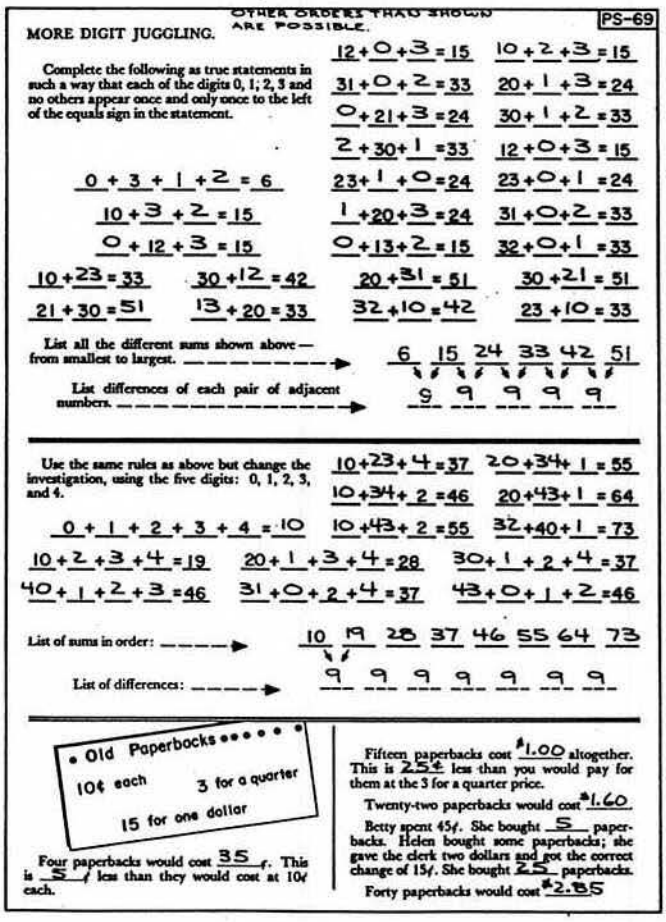
He can then turn them over and say something about adding feet—½ foot plus ¼ foot equals ¾ feet.

$$\boxed{\tfrac{1}{2}} + \boxed{\tfrac{1}{4}} = \boxed{\tfrac{3}{4}}$$

We strongly suggest that you or your class make such a set of cards. They will be useful again and again as you discuss fractions.

You might like to know that *Iso* is an abbreviation for *isomorphic*. Mathematicians would refer to the two parts of one of Ben's decks as isomorphic systems. This means that when you've done an addition or subtraction example no one can tell which part of the card you used—with the proper units the two answers are the same.

Practice sheet 69 begins a three-page study of *Digit Juggling* that culminates in a revealing problem on practice sheet 71.



OTHER ORDERS THAN SHOWN ARE POSSIBLE.

PS-69

MORE DIGIT JUGGLING.

Complete the following as true statements in such a way that each of the digits 0, 1, 2, 3 and no others appear once and only once to the left of the equals sign in the statement.

12+0+3=15 10+2+3=15
31+0+2=33 20+1+3=24
0+21+3=24 30+1+2=33
2+30+1=33 12+0+3=15
0+3+1+2=6 23+1+0=24 23+0+1=24
10+3+2=15 1+20+3=24 31+0+2=33
0+12+3=15 0+13+2=15 32+0+1=33
10+23=33 30+12=42 20+31=51 30+21=51
21+30=51 13+20=33 32+10=42 23+10=33

List all the different sums shown above — from smallest to largest. → 6 15 24 33 42 51

List differences of each pair of adjacent numbers. → 9 9 9 9 9

Use the same rules as above but change the investigation, using the five digits: 0, 1, 2, 3, and 4.

10+23+4=37 20+34+1=55
10+34+2=46 20+43+1=64
0+1+2+3+4=10 10+43+2=55 32+40+1=73
10+2+3+4=19 20+1+3+4=28 30+1+2+4=37
40+1+2+3=46 31+0+2+4=37 43+0+1+2=46

List of sums in order: → 10 19 28 37 46 55 64 73

List of differences: → 9 9 9 9 9 9 9

• Old Paperbacks • • • • •
10¢ each 3 for a quarter
15 for one dollar

Four paperbacks would cost 35¢. This is 5¢ less than they would cost at 10¢ each.

Fifteen paperbacks cost $1.00 altogether. This is 25¢ less than you would pay for them at the 3 for a quarter price.

Twenty-two paperbacks would cost $1.60

Betty spent 45¢. She bought 5 paperbacks. Helen bought some paperbacks; she gave the clerk two dollars and got the correct change of 15¢. She bought 25 paperbacks.

Forty paperbacks would cost $2.85

Ruler Iso Cards

"We might call this the Arithmetic of Iso Cards," Ben explained, "because Iso Cards are what I call them. You too can play this game."

	a.	b.	c.	d.
in.	12	6	4	3
ft.	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

	e.	f.	g.	h.	i.	j.	k.	l.	m.	n.
	8	9	2	10	1	18	24	15	21	7
	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{12}$	$1\frac{1}{2}$	2	$1\frac{1}{4}$	$1\frac{3}{4}$	$\frac{7}{12}$

o.

3	+	3	=	6
$\frac{1}{4}$	+	$\frac{1}{4}$	=	$\frac{1}{2}$

p.

8	+	4	=	12
$\frac{2}{3}$	+	$\frac{1}{3}$	=	1

q.

2	+	2	+	2	=	6
$\frac{1}{6}$	+	$\frac{1}{6}$	+	$\frac{1}{6}$	=	$\frac{1}{2}$

r.

12	−	4	=	8
1	−	$\frac{1}{3}$	=	$\frac{2}{3}$

s.

9	−	6	=	3
$\frac{3}{4}$	−	$\frac{1}{2}$	=	$\frac{1}{4}$

t.

6	+	6	+	6	=	18
$\frac{1}{2}$	+	$\frac{1}{2}$	+	$\frac{1}{2}$	=	$1\frac{1}{2}$

u.

4	+	3	=	7
$\frac{1}{3}$	+	$\frac{1}{4}$	=	$\frac{7}{12}$

v.

2	−	1	=	1
$\frac{1}{6}$	−	$\frac{1}{12}$	=	$\frac{1}{12}$

w.

4	+	4	+	4	=	12
$\frac{1}{3}$	+	$\frac{1}{3}$	+	$\frac{1}{3}$	=	1

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

A.

$\frac{1}{3}$	+	$\frac{1}{3}$	=	$\frac{2}{3}$

B.

$\frac{1}{4}$	+	$\frac{1}{2}$	=	$\frac{3}{4}$

C.

$\frac{1}{4}$	+	$\frac{1}{4}$	+	$\frac{1}{4}$	=	$\frac{3}{4}$

D.

$\frac{1}{4}$	+	$\frac{3}{4}$	=	1

E.

2	−	$\frac{1}{2}$	=	$1\frac{1}{2}$

F.

$\frac{1}{2}$	+	$\frac{1}{4}$	+	$\frac{1}{4}$	=	1

G.

$\frac{7}{12}$	−	$\frac{1}{2}$	=	$\frac{1}{12}$

H.

$\frac{1}{6}$	+	$\frac{1}{3}$	=	$\frac{1}{2}$

I.

$\frac{1}{3}$	+	1	+	$\frac{2}{3}$	=	2

J.

$\frac{3}{4}$	+	$\frac{1}{2}$	=	$1\frac{1}{4}$

K.

$\frac{1}{3}$	−	$\frac{1}{4}$	=	$\frac{1}{12}$

L.

$\frac{2}{3}$	+	$\frac{2}{3}$	+	$\frac{2}{3}$	=	2

Tasks: In exercises a-n *pupils complete the Iso Cards. In exercises* o-w *pupils complete the Iso Cards to make true sentences. In exercises A-L they complete the bottom portion of Iso Cards to make true sentences.*

Purpose: To continue the study of addition and subtraction of fractions.

Unifying Ideas: Addition and subtraction; measurement.

The Lesson: Ben presents a page of Iso Card arithmetic, in which all statements have to do with adding or subtracting lengths. He reports each addition or subtraction in terms of both inches and feet.

Pupils will have little trouble in writing:

$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$$

when they know that there are 4 inches in one-third of a foot and 3 inches in one-fourth of a foot. They know that $4+3=7$; and they know that Ben refers to 7 inches as $\frac{7}{12}$ ft.

Nor will they be at all bewildered by the problems at the bottom of pupil page 70, in which the side of the cards noting number of inches is not shown. They will note the number of inches, do the adding or subtracting in their heads or on scratch paper, and then convert the answer from a number of inches to a number of feet.

Although this may seem clumsy, a pupil must first be able to think his way through a problem before he is given rules to follow. The pupils must have at least one method, clumsy though it may be, that they can use to think through to a sensible result. This applies to many areas besides mathematics.

Further, many pupils will begin inventing shortcuts to avoid having to translate fractions of a foot into inches. Every such invention is a solid foundation on which to build an understanding of more formal manipulations to come.

Other systems of measurement will be explored in the pupil pages that follow. Eventually, pupils will learn a more general method of adding and subtracting fractions.

Practice sheet 70 presents more *Digit Juggling*. In exercise A the differences turn out to be 9's or multiples of 9. In exercise B all differences are 9's.

You might wish to know that when the digits selected are sequential—such as 4, 5, 6, 7, and 8—the differences will all be 9. When they are not sequential, the differences will be 9 or multiples of 9.

You can anticipate these results by noticing the differences between the digits in the sequence to be explored:

4 5 6 7 8	0 1 5 7 9
1 1 1 1	1 4 2 2
$9\times1=9$	$9\times1=9$
	$9\times4=36$
	$9\times2=18$

A. More Digit Juggling — PS-70

OTHER ORDERS THAN SHOWN ARE POSSIBLE.

Complete the following as true statements in such a way that each of the digits 0, 1, 5, 7, 9 and no others appear once and only once to the left of the equals sign in the statement.

	1 + 57 + 90 = 148	71 + 50 + 9 = 130
	15 + 7 + 90 = 112	70 + 15 + 9 = 94
	15 + 79 + 0 = 94	15 + 97 + 0 = 112
	1 + 79 + 50 = 130	79 + 51 + 0 = 130
	17 + 95 + 0 = 112	90 + 51 + 7 = 148
0 + 1 + 5 + 7 + 9 = 22	17 + 9 + 50 = 76	90 + 75 + 1 = 166
10 + 5 + 7 + 9 = 31	90 + 1 + 5 + 7 = 103	51 + 7 + 0 + 9 = 67
50 + 1 + 7 + 9 = 67	57 + 1 + 9 + 0 = 67	15 + 7 + 0 + 9 = 31
70 + 1 + 5 + 9 = 85	75 + 1 + 9 + 0 = 85	97 + 0 + 1 + 5 = 103

List the sums in order. 22 31 67 76 85 94 103 112 130 148 166

List the differences. 9 36 9 9 9 9 9 18 18 18

B.

Extend the investigation using a different set of 5 digits: 4, 5, 6, 7, and 8

4 + 57 + 68 = 129	4 + 75 + 86 = 165
5 + 67 + 48 = 120	5 + 74 + 86 = 165
6 + 87 + 54 = 147	6 + 78 + 54 = 138
7 + 54 + 68 = 129	7 + 85 + 64 = 156
8 + 46 + 57 = 111	8 + 75 + 64 = 147
4 + 5 + 6 + 7 + 8 = 30	
46 + 5 + 7 + 8 = 66	8 + 7 + 56 + 4 = 75
54 + 6 + 7 + 8 = 75	64 + 5 + 7 + 8 = 84
8 + 76 + 5 + 4 = 93	8 + 45 + 6 + 7 = 66
	74 + 5 + 6 + 8 = 93
	84 + 5 + 6 + 7 = 102
	4 + 5 + 67 + 8 = 84

List the sums in order. 30 66 75 84 93 102 111 120 129 138 147 156 165

List the differences. 36 9 9 9 9 9 9 9 9 9 9 9

C.

Jean paid $1.80 for three yards of cloth. This would be at the rate of 60¢ per yard.

She cut the material in 4 pieces, all the same size. The cost of each piece was 45¢. Each was 27 in. long.

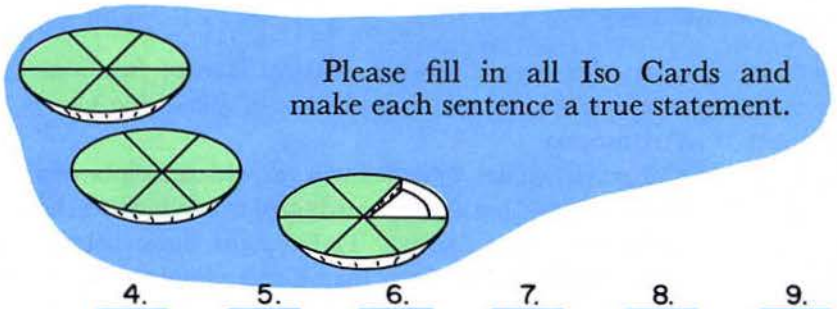
D.

A map reads, "1 inch = 50 miles." Two inches on the map would be 100 miles. Five inches on the map would be 250 miles.

York and Alva are 150 miles apart. On the map they are 3 inches apart.

A thousand miles would be shown on the map by 20 inches.

"I made other sets of cards," Ben explained, "when I thought of other things. Can you complete this new set of Iso Cards?"



Please fill in all Iso Cards and make each sentence a true statement.

	1.	2.	3.
pieces of pie	1	6	3
number of pies	$\frac{1}{6}$	1	$\frac{1}{2}$

	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
	12	2	5	9	4	7	14	22	11	31
	2	$\frac{1}{3}$	$\frac{5}{6}$	$1\frac{1}{2}$	$\frac{2}{3}$	$1\frac{1}{6}$	$2\frac{1}{3}$	$3\frac{2}{3}$	$1\frac{5}{6}$	$5\frac{1}{6}$

14.

3	+	3	=	6
$\frac{1}{2}$	+	$\frac{1}{2}$	=	1

15.

4	+	2	=	6
$\frac{2}{3}$	+	$\frac{1}{3}$	=	1

16.

1	+	2	+	3	=	6
$\frac{1}{6}$	+	$\frac{1}{3}$	+	$\frac{1}{2}$	=	1

17.

6	−	2	=	4
1	−	$\frac{1}{3}$	=	$\frac{2}{3}$

18.

9	−	4	=	5
$1\frac{1}{2}$	−	$\frac{2}{3}$	=	$\frac{5}{6}$

19.

5	+	3	+	4	=	12
$\frac{5}{6}$	+	$\frac{1}{2}$	+	$\frac{2}{3}$	=	2

20.

8	+	11	=	19
$1\frac{1}{3}$	+	$1\frac{5}{6}$	=	$3\frac{1}{6}$

21.

10	−	2	=	8
$1\frac{2}{3}$	−	$\frac{1}{3}$	=	$1\frac{1}{3}$

22.

9	+	16	+	1	=	26
$1\frac{1}{2}$	+	$2\frac{2}{3}$	+	$\frac{1}{6}$	=	$4\frac{1}{3}$

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

23.

$\frac{1}{3}$	+	$\frac{1}{3}$	=	$\frac{2}{3}$

24.

1	−	$\frac{1}{6}$	=	$\frac{5}{6}$

25.

$\frac{1}{2}$	+	$\frac{1}{2}$	+	$\frac{1}{2}$	=	$1\frac{1}{2}$

26.

$2\frac{1}{2}$	−	$\frac{2}{3}$	=	$1\frac{5}{6}$

27.

$\frac{5}{6}$	+	$1\frac{5}{6}$	=	$2\frac{2}{3}$

28.

$1\frac{1}{3}$	+	$\frac{5}{6}$	+	$1\frac{1}{2}$	=	$3\frac{2}{3}$

29.

$1\frac{1}{6}$	−	$\frac{1}{3}$	=	$\frac{5}{6}$

30.

$\frac{1}{3}$	+	$\frac{1}{2}$	=	$\frac{5}{6}$

31.

$\frac{2}{3}$	+	$\frac{2}{3}$	+	$\frac{2}{3}$	=	2

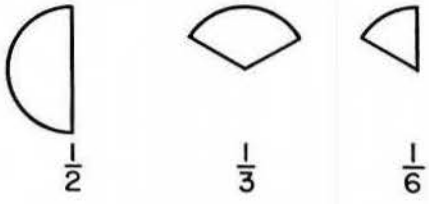
Tasks: In exercises 1-13 pupils complete each Iso Card. In exercises 14-22 pupils complete the Iso Cards to make true statements. In exercises 23-31 they complete the bottom sections of Iso Cards to make true statements.

Purpose: To explore a rationale for adding and subtracting fractions. To develop computational skills with fractions.

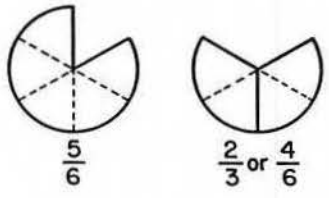
Unifying Ideas: Number and counting; numeration; addition and subtraction.

The Lesson: Ben introduces his second set of Iso Cards. If the crust of a pie is marked for cutting into six pieces of equal size, in how many ways can one use these marks to cut the pie into parts of equal size? Ben asked the same question about the ruler and its inch markings. The pie can be cut in these ways:

Ben would refer to these cuts as $\overline{2}$, $\overline{3}$, and $\overline{6}$. This gives him labels for the following:



How can he label other pieces that could be cut out using only the markings on the crust? How can each of these pieces be made up of the fewest number of one of the pieces above?



In the latter case, he chooses two-thirds because it requires fewer pieces.

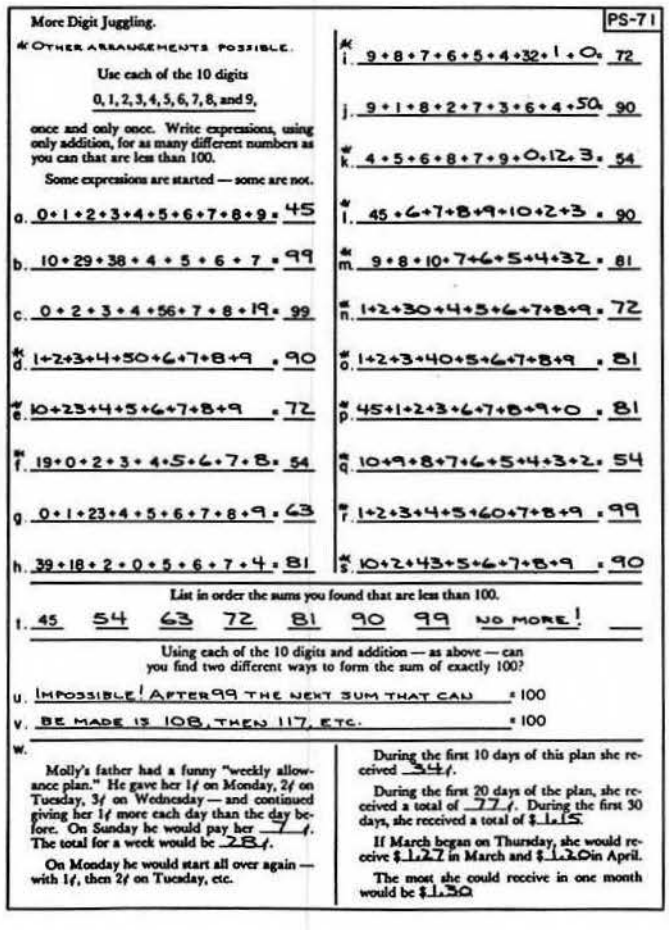
Practice sheet 71 summarizes the digit juggling pages. Some pupils may have noticed that as a digit is moved from the 1's place to the 10's place its value increases by 9 or by a multiple of 9. In the units place, 1 contributes 1; in the 10's place, it contributes 10—or 9 more. Similarly, 7 contributes 63 more in the 10's place than in the units place, and 63 is a multiple of 9. This sheds considerable light on the problem presented on practice sheet 71. We note that the sum of the digits 0 through 9:

$$0+1+2+3+4+5+6+7+8+9$$

is a multiple of 9—45. Suppose that we combine two of the digits in this list into a two-digit numeral, say 67. We in effect simply move the 6 from the units place to the tens place (since the 7 remains in the units place).

$$0+1+2+3+4+5+\underline{60}+7+8+9$$

We should be able to predict that this new sum will be a multiple of 9, since 60 is greater than 6 by a multiple of 9. This should also help to explain why the answer to u and v is that there is no such expression whose sum is 100.



More Digit Juggling. PS-71

* Other arrangements possible.

Use each of the 10 digits
0, 1, 2, 3, 4, 5, 6, 7, 8, and 9,
once and only once. Write expressions, using only addition, for as many different numbers as you can that are less than 100.

Some expressions are started — some are not.

a. 0+1+2+3+4+5+6+7+8+9 = 45
b. 10+29+38+4+5+6+7 = 99
c. 0+2+3+4+56+7+8+19 = 99
* d. 1+2+3+4+50+6+7+8+9 = 90
* e. 10+23+4+5+6+7+8+9 = 72
* f. 19+0+2+3+4+5+6+7+8 = 54
g. 0+1+23+4+5+6+7+8+9 = 63
h. 39+18+2+0+5+6+7+4 = 81

* i. 9+8+7+6+5+4+32+1+0 = 72
j. 9+1+8+2+7+3+6+4+50 = 90
* k. 4+5+6+8+7+9+0+12+3 = 54
* l. 45+6+7+8+9+10+2+3 = 90
* m. 9+8+10+7+6+5+4+32 = 81
* n. 1+2+30+4+5+6+7+8+9 = 72
* o. 1+2+3+40+5+6+7+8+9 = 81
* p. 45+1+2+3+6+7+8+9+0 = 81
* q. 10+9+8+7+6+5+4+3+2 = 54
* r. 1+2+3+4+5+60+7+8+9 = 99
* s. 10+2+43+5+6+7+8+9 = 90

List in order the sums you found that are less than 100.

1. 45 54 63 72 81 90 99 NO MORE!

Using each of the 10 digits and addition — as above — can you find two different ways to form the sum of exactly 100?

u. IMPOSSIBLE! AFTER 99 THE NEXT SUM THAT CAN = 100
v. BE MADE IS 108, THEN 117, ETC. = 100

w. Molly's father had a funny "weekly allowance plan." He gave her 1¢ on Monday, 2¢ on Tuesday, 3¢ on Wednesday — and continued giving her 1¢ more each day than the day before. On Sunday he would pay her 7¢. The total for a week would be 28¢.

On Monday he would start all over again — with 1¢, then 2¢ on Tuesday, etc.

During the first 10 days of this plan she received 55¢.

During the first 20 days of the plan, she received a total of 77¢. During the first 30 days, she received a total of $1.15.

If March began on Thursday, she would receive $1.27 in March and $1.20 in April.

The most she could receive in one month would be $1.30

Use Ben's plan to fill in the tinted blocks so that each card is correct and each sentence is a true statement.

	a.	b.	c.
qt.	4	2	1
gal.	1	$\frac{1}{2}$	$\frac{1}{4}$

d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
5	6	3	12	7	9	14	11	16	40
$1\frac{1}{4}$	$1\frac{1}{2}$	$\frac{3}{4}$	3	$1\frac{3}{4}$	$2\frac{1}{4}$	$3\frac{1}{2}$	$2\frac{3}{4}$	4	10

n.

1	+	1	=	2
$\frac{1}{4}$	+	$\frac{1}{4}$	=	$\frac{1}{2}$

o.

4	−	1	=	3
1	−	$\frac{1}{4}$	=	$\frac{3}{4}$

p.

3	+	3	+	3	=	9
$\frac{3}{4}$	+	$\frac{3}{4}$	+	$\frac{3}{4}$	=	$2\frac{1}{4}$

q.

1	+	3	=	4
$\frac{1}{4}$	+	$\frac{3}{4}$	=	1

r.

3	−	2	=	1
$\frac{3}{4}$	−	$\frac{1}{2}$	=	$\frac{1}{4}$

s.

1	+	2	+	3	=	6
$\frac{1}{4}$	+	$\frac{1}{2}$	+	$\frac{3}{4}$	=	$1\frac{1}{2}$

t.

6	+	6	=	12
$1\frac{1}{2}$	+	$1\frac{1}{2}$	=	3

u.

8	−	3	=	5
2	−	$\frac{3}{4}$	=	$1\frac{1}{4}$

v.

7	+	8	+	2	=	17
$1\frac{3}{4}$	+	2	+	$\frac{1}{2}$	=	$4\frac{1}{4}$

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

A.

$\frac{1}{2}$	+	$1\frac{1}{2}$	=	2

B.

$\frac{1}{2}$	−	$\frac{1}{4}$	=	$\frac{1}{4}$

C.

$1\frac{1}{2}$	+	$\frac{3}{4}$	+	$\frac{3}{4}$	=	3

D.

2	+	3	=	5

E.

$\frac{3}{4}$	+	$\frac{3}{4}$	=	$1\frac{1}{2}$

F.

3	+	$2\frac{1}{2}$	+	$1\frac{1}{2}$	=	7

G.

$1\frac{1}{4}$	+	$1\frac{1}{4}$	=	$2\frac{1}{2}$

H.

$3\frac{1}{2}$	−	$\frac{3}{4}$	=	$2\frac{3}{4}$

I.

$\frac{1}{4}$	+	$\frac{1}{4}$	+	$1\frac{1}{4}$	=	$1\frac{3}{4}$

J.

$4\frac{1}{2}$	+	$4\frac{1}{2}$	=	9

K.

$10\frac{1}{4}$	+	3	=	$13\frac{1}{4}$

L.

$2\frac{1}{2}$	+	$2\frac{1}{2}$	+	$2\frac{1}{2}$	=	$7\frac{1}{2}$

Tasks: *In exercises* a-m *pupils use Ben's plan to complete each Iso Card. In exercises* n-v *they complete the Iso Cards to make true sentences. In exercises A-L they complete the bottom sections of Iso Cards to make true sentences.*

Purpose: To continue exploring a rationale for adding and subtracting fractions. To develop computational skills with fractions.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

The Lesson: This set of Iso Cards is much easier to work with than the two sets introduced in previous pupil pages. Perhaps many pupils can work through this pupil page on their own, relieving you to work with those who have not yet fully understood Ben's Iso Cards and the arithmetic he does with those cards.

Some pupils will notice something rather significant about these sets of cards. Consider three examples—with three different sets of cards:

Ruler	Pies	Gallons
in. 6 + 6 = 12	pcs. 3 + 3 = 6	qt. 2 + 2 = 4
ft. $\frac{1}{2}$ + $\frac{1}{2}$ = 1	pies $\frac{1}{2}$ + $\frac{1}{2}$ = 1	gal. $\frac{1}{2}$ + $\frac{1}{2}$ = 1

The arithmetic of inches, of pieces of pie, and of quarts makes three different statements—6+6=12, 3+3=6, and 2+2=4—while Ben's new method of reporting makes one statement three times:

$$\frac{1}{2} + \frac{1}{2} = 1$$

This is hardly surprising. Half an hour plus half an hour is one hour; half a candy bar plus half a candy bar is one bar. Try another group:

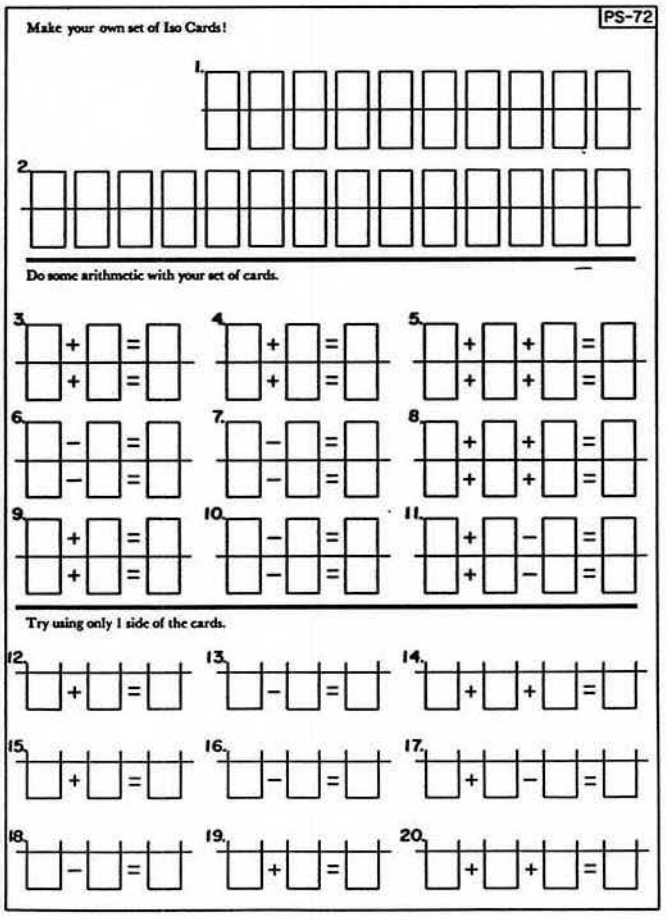
Ruler	Pies	Gallons
in. 18 + 18 = 36	pcs. 9 + 9 = 18	qt. 6 + 6 = 12
ft. $1\frac{1}{2}$ + $1\frac{1}{2}$ = 3	pies $1\frac{1}{2}$ + $1\frac{1}{2}$ = 3	gal. $1\frac{1}{2}$ + $1\frac{1}{2}$ = 3

A dollar and a half plus a dollar and a half is three dollars; a dozen and a half plus a dozen and a half is three dozen.

In this way, the more general idea of fractions emerges from the developments of Ben's Iso Card Arithmetic.

Practice sheet 72 is a form on which pupils can practice building a set of cards and working with the arithmetic that follows. They might build sets of cards based on baseball teams and players, nickels and pennies, pies and pieces with 8 pieces per pie, etc. Or, they might explore further the three sets of Iso Cards already considered. Or, you may hold this practice sheet for exploration of other sets of Iso Cards yet to be developed.

The practice sheet has a section in which only one side of the card is shown. This challenges pupils to find shortcuts rather than always translating into inches or pieces of pie or quarts. Each shortcut is, in actuality, a step toward understanding that fractions can be treated as abstractions. However, this understanding will be built on the most permanent foundation if pupils invent their own shortcuts.



Use Ben's plan to fill in the tinted blocks so that each card is correct and each sentence is a true statement.

cookies	18	9	6
boxes	1	$\frac{1}{2}$	$\frac{1}{3}$

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1	2	3	5	4	12	17	15	8	24
$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{17}{18}$	$\frac{5}{6}$	$\frac{4}{9}$	$1\frac{1}{3}$

11.

18	+	18	=	36
1	+	1	=	2

12.

2	+	6	=	8
$\frac{1}{9}$	+	$\frac{1}{3}$	=	$\frac{4}{9}$

13.

2	+	2	+	2	=	6
$\frac{1}{9}$	+	$\frac{1}{9}$	+	$\frac{1}{9}$	=	$\frac{1}{3}$

14.

1	+	1	=	2
$\frac{1}{18}$	+	$\frac{1}{18}$	=	$\frac{1}{9}$

15.

7	+	8	=	15
$\frac{7}{18}$	+	$\frac{4}{9}$	=	$\frac{5}{6}$

16.

4	+	4	+	4	=	12
$\frac{2}{9}$	+	$\frac{2}{9}$	+	$\frac{2}{9}$	=	$\frac{2}{3}$

17.

20	−	8	=	12
$1\frac{1}{9}$	−	$\frac{4}{9}$	=	$\frac{2}{3}$

18.

17	−	9	=	8
$\frac{17}{18}$	−	$\frac{1}{2}$	=	$\frac{4}{9}$

19.

1	+	2	+	3	=	6
$\frac{1}{18}$	+	$\frac{1}{9}$	+	$\frac{1}{6}$	=	$\frac{1}{3}$

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

20.

$\frac{1}{2}$	+	$\frac{1}{3}$	=	$\frac{5}{6}$

21.

$\frac{1}{6}$	+	$\frac{1}{9}$	=	$\frac{5}{18}$

22.

$\frac{5}{18}$	+	$\frac{6}{18}$	+	$\frac{7}{18}$	=	1

23.

$\frac{2}{9}$	+	$\frac{1}{2}$	=	$\frac{13}{18}$

24.

$\frac{5}{9}$	−	$\frac{1}{6}$	=	$\frac{7}{18}$

25.

$\frac{1}{18}$	+	$\frac{1}{18}$	+	$\frac{1}{18}$	=	$\frac{1}{6}$

26.

$1\frac{1}{3}$	−	$\frac{1}{2}$	=	$\frac{5}{6}$

27.

$1\frac{2}{9}$	−	$\frac{13}{18}$	=	$\frac{1}{2}$

28.

$\frac{1}{3}$	+	$\frac{1}{3}$	+	$\frac{1}{3}$	=	1

29.

$2\frac{1}{9}$	−	$\frac{5}{18}$	=	$1\frac{5}{6}$

30.

$3\frac{1}{3}$	−	$1\frac{2}{3}$	=	$1\frac{2}{3}$

31.

$1\frac{2}{3}$	+	$1\frac{2}{3}$	+	$1\frac{2}{3}$	=	5

73 Boxes of cookies (eighteenths)

Tasks: *In exercises 1-10 pupils use Ben's plan to complete each Iso Card. In exercises 11-19 pupils complete the Iso Cards to make true sentences. In exercises 20-31 they complete the bottom sections of Iso Cards to make true statements.*

Purpose: To continue exploring a rationale for adding and subtracting fractions. To develop computational skills with fractions.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

The Lesson: This set of Ben's Iso Cards starts with a package of cookies as the basic unit. A package contains 18 cookies. In how many ways can 18 cookies be divided into groups of the same size?

2 groups, 9 in each . . . $\frac{1}{2}$ package

3 groups, 6 in each . . . $\frac{1}{3}$ package

6 groups, 3 in each . . . $\frac{1}{6}$ package

9 groups, 2 in each . . . $\frac{1}{9}$ package

18 groups, 1 in each . . . $\frac{1}{18}$ package

How could we report 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, and 17 cookies in a group? There are complicated ways:

4 cookies: $\frac{1}{6}$ pk. + $\frac{1}{18}$ pk.

5 cookies: $\frac{1}{6}$ pk. + $\frac{1}{9}$ pk.

7 cookies: $\frac{1}{3}$ pk. + $\frac{1}{18}$ pk., etc.

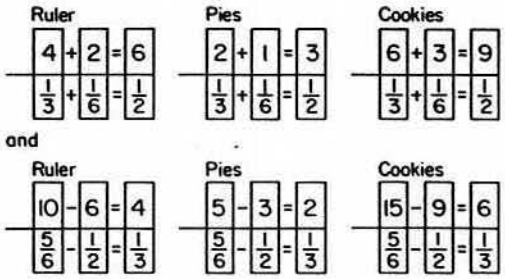
But we know Ben will improve on these by finding the largest group that could be repeated to specify a certain number of cookies:

4 cookies: $\frac{4}{18}$ or $\frac{2}{9}$ (and he chooses the latter)

5 cookies: $\frac{5}{18}$ (and he can't use larger groups)

etc.

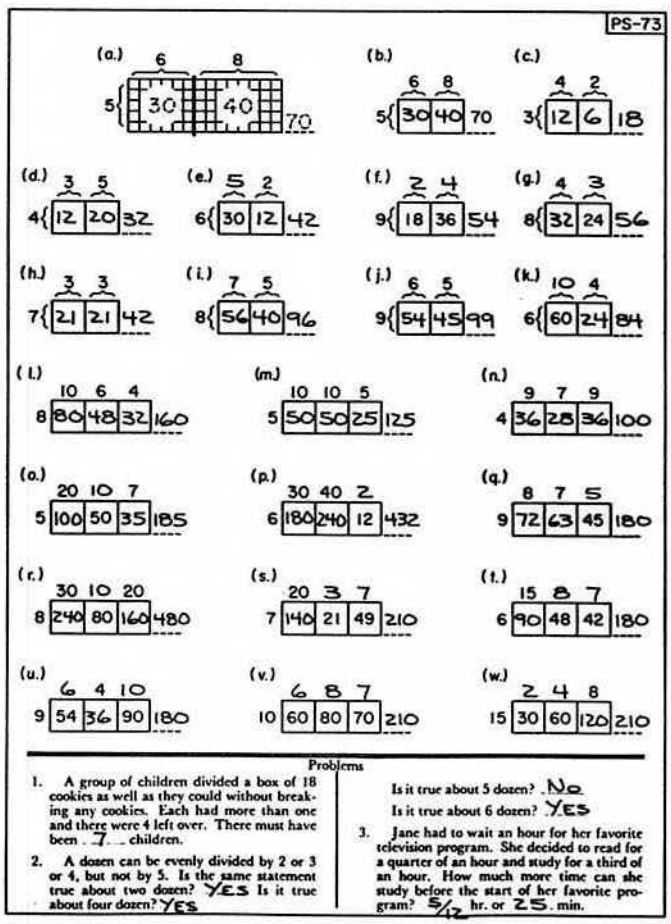
After working with the set of cards on pupil page 73, present special examples from the ruler, pies, and cookie sets:

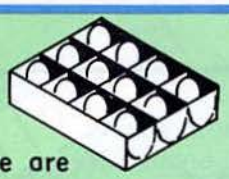


Can you find more examples using these sets of cards? There are many—and the search is rewarding.

Practice sheet 73 is preparation for a new version of *Punch Card Arithmetic* (see practice sheets 74-78, 80, and 81). This series will explore the division algorithm from yet another angle—and will provide much practice along the way in addition, subtraction, multiplication, and division computations.

The development begins with the suggestion of a way to report certain rectangular arrangements of tiles or blocks or boxes on a piece of graph paper. The example and the clues given ought to enable the pupils to work on their own, once underway.


PS-73
(a.) 6 8 | 5 | 30 40 | 70
(b.) 6 8 | 5 | 30 40 | 70
(c.) 4 2 | 3 | 12 6 | 18
(d.) 3 5 | 4 | 12 20 | 32
(e.) 5 2 | 6 | 30 12 | 42
(f.) 2 4 | 9 | 18 36 | 54
(g.) 4 3 | 8 | 32 24 | 56
(h.) 3 3 | 7 | 21 21 | 42
(i.) 7 5 | 8 | 56 40 | 96
(j.) 6 5 | 9 | 54 45 | 99
(k.) 10 4 | 6 | 60 24 | 84
(l.) 10 6 4 | 8 | 80 48 32 | 160
(m.) 10 10 5 | 5 | 50 50 25 | 125
(n.) 9 7 9 | 4 | 36 28 36 | 100
(o.) 20 10 7 | 5 | 100 50 35 | 185
(p.) 30 40 2 | 6 | 180 240 12 | 432
(q.) 8 7 5 | 9 | 72 63 45 | 180
(r.) 30 10 20 | 8 | 240 80 160 | 480
(s.) 20 3 7 | 7 | 140 21 49 | 210
(t.) 15 8 7 | 6 | 90 48 42 | 180
(u.) 6 4 10 | 9 | 54 36 90 | 180
(v.) 6 8 7 | 10 | 60 80 70 | 210
(w.) 2 4 8 | 15 | 30 60 120 | 210
Problems
1. A group of children divided a box of 18 cookies as well as they could without breaking any cookies. Each had more than one and there were 4 left over. There must have been 7 children.
2. A dozen can be evenly divided by 2 or 3 or 4, but not by 5. Is the same statement true about two dozen? YES Is it true about four dozen? YES
Is it true about 5 dozen? No
Is it true about 6 dozen? YES
3. Jane had to wait an hour for her favorite television program. She decided to read for a quarter of an hour and study for a third of an hour. How much more time can she study before the start of her favorite program? 5/12 hr. or 25 min.



There are 12 eggs in 1 dozen.

	I.	2.	3.	4.	5.	6.	7.
eggs	6	12	4	8	3	9	10
doz.	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{6}$

A. $2 + 4 = 6$

$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

B. $7 - 3 = 4$

$\frac{7}{12} - \frac{1}{4} = \frac{1}{3}$

C. $2 + 3 + 6 = 11$

$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$

D. $18 - 8 = 10$

$1\frac{1}{2} - \frac{2}{3} = \frac{5}{6}$

E. $6 + 10 = 16$

$\frac{1}{2} + \frac{5}{6} = 1\frac{1}{3}$

F. $6 + 8 + 9 = 23$

$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = 1\frac{11}{12}$

G. $3 - 1\frac{5}{6} = 1\frac{1}{6}$

H. $2\frac{1}{3} - \frac{3}{4} = 1\frac{7}{12}$

I. $1\frac{1}{2} + 2\frac{1}{4} + 3\frac{1}{2} = 7\frac{1}{4}$

J. $\frac{1}{12} + \frac{7}{12} = \frac{2}{3}$

K. $\frac{2}{3} + \frac{5}{12} = 1\frac{1}{12}$

L. $3\frac{1}{4} + \frac{2}{3} + 1\frac{2}{3} = 5\frac{7}{12}$

	I.	II.	III.	IV.	V.	VI.	VII.
hours	24	12	8	6	3	2	1
days	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$

M. $4 + 8 = 12$

$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

N. $3 + 9 = 12$

$\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

O. $10 + 10 + 10 = 30$

$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} = 1\frac{1}{4}$

P. $8 + 12 = 20$

$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

Q. $9 - 8 = 1$

$\frac{3}{8} - \frac{1}{3} = \frac{1}{24}$

R. $5 + 1 + 1 = 7$

$\frac{5}{24} + \frac{1}{24} + \frac{1}{24} = \frac{7}{24}$

S. $\frac{3}{4} + \frac{11}{12} = 1\frac{2}{3}$

T. $\frac{5}{8} + \frac{1}{4} = \frac{7}{8}$

U. $\frac{1}{8} + \frac{1}{4} + \frac{3}{8} = \frac{3}{4}$

V. $\frac{7}{24} + \frac{11}{24} = \frac{3}{4}$

W. $\frac{3}{8} + \frac{1}{6} = \frac{13}{24}$

X. $\frac{5}{6} + \frac{1}{3} + \frac{1}{2} = 1\frac{2}{3}$

74 Dozens (twelfths) and days and hours (twenty-fourths)

Tasks: In exercises 1-7 and I-VII pupils complete the Iso Cards. In exercises A-F and M-R pupils complete the Iso Cards to make true sentences. In exercises G-L and S-X they complete the bottom of Iso Cards to make true sentences.

Purpose: To develop computational skills for adding and subtracting fractions.

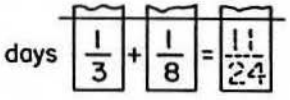
Unifying Ideas: Addition and subtraction.

The Lesson: As we consider the first set of Ben's cards on this pupil page—with the basic unit a dozen eggs—we notice something quite interesting. As we build this set, we realize that it is very much like the first set Ben built. In fact, it is identical. We can use the old set just as well for dozens of eggs and single eggs as for feet and inches. This perhaps unexpected event will contribute another bit toward understanding fractions.

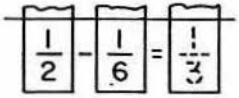
The second set of cards on this pupil page presents days as the basic unit with hours on the other half of the card. Nothing very new arises. That 24 is divisible by 8 leads to some new expressions:

3 hours . . . ⅛ of a day
9 hours . . . ⅜ of a day
6 hours . . . could be referred to as ⅜ of a day, but Ben's rule leads him to choose ¼

In both developments on pupil page 74, you will notice fewer examples displaying both parts of Ben's cards. We have done this to encourage the invention of shortcuts to avoid having to convert to the smaller unit each time. Often there is no alternative, for example:

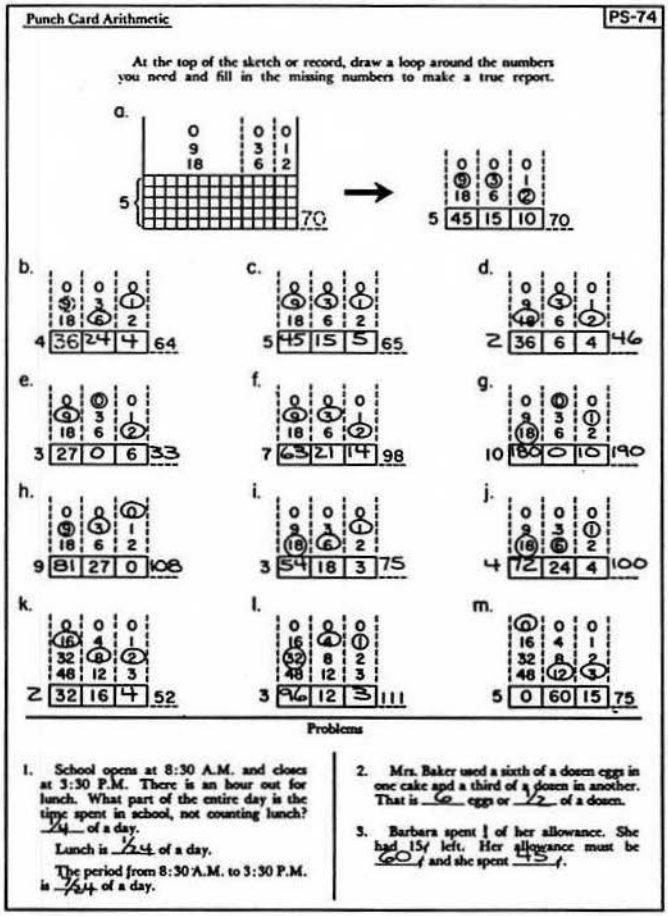


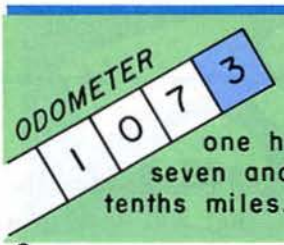
There is an alternative, however, for:



For this problem, instead of thinking about hours and days, we could think about pies cut in 6 pieces. (Finding this kind of shortcut will later be referred to as "finding the least common denominator.")

Practice sheet 74 is a version of *Punch Card Arithmetic.* Instead of writing numbers, we punch out a combination the sum of whose products with the number on the left is the number on the right.


Punch Card Arithmetic
PS-74
At the top of the sketch or record, draw a loop around the numbers you need and fill in the missing numbers to make a true report.
a.
5 45 15 10 70
b.
4 36 24 4 64
c.
5 45 15 5 65
d.
2 36 6 4 46
e.
3 27 0 6 33
f.
7 63 21 14 98
g.
10 180 0 10 190
h.
9 81 27 0 108
i.
3 54 18 3 75
j.
4 72 24 4 100
k.
2 32 16 4 52
l.
3 96 12 3 111
m.
5 0 60 15 75
Problems
1. School opens at 8:30 A.M. and closes at 3:30 P.M. There is an hour out for lunch. What part of the entire day is the time spent in school, not counting lunch? 1/4 of a day.
Lunch is 1/24 of a day.
The period from 8:30 A.M. to 3:30 P.M. is 7/24 of a day.
2. Mrs. Baker used a sixth of a dozen eggs in one cake and a third of a dozen in another. That is 2 eggs or 1/2 of a dozen.
3. Barbara spent ¼ of her allowance. She had 15¢ left. Her allowance must be 60¢ and she spent 45¢.



one hundred seven and three tenths miles.

	1.	2.	3.	4.	5.	6.	7.
tenths	10	1	5	2	4	3	12
miles	1	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{10}$	$1\frac{1}{5}$

8.

1	+	1	=	2
$\frac{1}{10}$	+	$\frac{1}{10}$	=	$\frac{1}{5}$

9.

2	+	4	=	6
$\frac{1}{5}$	+	$\frac{2}{5}$	=	$\frac{3}{5}$

10.

2	+	2	+	1	=	5
$\frac{1}{5}$	+	$\frac{1}{5}$	+	$\frac{1}{10}$	=	$\frac{1}{2}$

11.

9	−	7	=	2
$\frac{9}{10}$	−	$\frac{7}{10}$	=	$\frac{1}{5}$

12.

5	+	6	=	11
$\frac{1}{2}$	+	$\frac{3}{5}$	=	$1\frac{1}{10}$

13.

4	+	6	+	7	=	17
$\frac{2}{5}$	+	$\frac{3}{5}$	+	$\frac{7}{10}$	=	$1\frac{7}{10}$

14.

$\frac{7}{10}$	+	$\frac{3}{10}$	=	1

15.

$2\frac{1}{2}$	−	$\frac{2}{5}$	=	$2\frac{1}{10}$

16.

$1\frac{3}{5}$	+	$\frac{3}{10}$	+	$1\frac{1}{2}$	=	$3\frac{2}{5}$

17.

$\frac{2}{5}$	−	$\frac{3}{10}$	=	$\frac{1}{10}$

18.

$\frac{9}{10}$	+	$1\frac{1}{10}$	=	2

19.

$7\frac{1}{10}$	+	$10\frac{9}{10}$	+	$4\frac{3}{5}$	=	$22\frac{3}{5}$

There are 16 ounces in a pound.

	20.	21.	22.	23.	24.	25.	26.
ozs.	8	4	16	2	1	6	12
lbs.	$\frac{1}{2}$	$\frac{1}{4}$	1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{3}{4}$

27.

1	+	3	=	4
$\frac{1}{16}$	+	$\frac{3}{16}$	=	$\frac{1}{4}$

28.

8	−	3	=	5
$\frac{1}{2}$	−	$\frac{3}{16}$	=	$\frac{5}{16}$

29.

2	+	4	+	2	=	8
$\frac{1}{8}$	+	$\frac{1}{4}$	+	$\frac{1}{8}$	=	$\frac{1}{2}$

30.

16	−	7	=	9
1	−	$\frac{7}{16}$	=	$\frac{9}{16}$

31.

2	+	8	=	10
$\frac{1}{8}$	+	$\frac{1}{2}$	=	$\frac{5}{8}$

32.

24	+	4	+	6	=	34
$1\frac{1}{2}$	+	$\frac{1}{4}$	+	$\frac{3}{8}$	=	$2\frac{1}{8}$

33.

$\frac{15}{16}$	+	$\frac{1}{16}$	=	1

34.

$\frac{7}{8}$	−	$\frac{1}{2}$	=	$\frac{3}{8}$

35.

$1\frac{3}{16}$	+	$2\frac{1}{4}$	+	$1\frac{5}{8}$	=	$5\frac{1}{16}$

36.

$\frac{1}{8}$	+	$\frac{1}{8}$	=	$\frac{1}{4}$

37.

$1\frac{3}{4}$	−	$\frac{7}{8}$	=	$\frac{7}{8}$

38.

$\frac{3}{8}$	+	$\frac{3}{16}$	+	$\frac{1}{2}$	=	$1\frac{1}{16}$

75 Miles and tenths (tenths) and pounds and ounces (sixteenths)

Tasks: *In exercises 1-7 and 20-26 pupils complete the Iso Cards. In exercises 8-13 and 27-32 pupils complete the Iso Cards to make true sentences. In exercises 14-19 and 33-38 they complete the bottom sections of Iso Cards to make true sentences.*

Purpose: To develop computational skills for adding and subtracting fractions.

Unifying Ideas: Addition and subtraction.

The Lesson: The first set of cards on this pupil page, suggested by the automobile odometer, is a forerunner of decimal fractions. But we are not there yet. So Ben writes

$$\frac{1}{5} \text{ rather than } \frac{2}{10}$$

etc.

None of the sets of cards considered thus far has included the notion of fifths, so examples from this first set adds something entirely new. But the other set—pounds and ounces—has much in common with gallons and quarts and hours and days, and something in common with feet and inches and dozens of eggs and single eggs. Here are some examples:

lbs. & oz.

14	−	4	=	10		
$\frac{7}{8}$	−	$\frac{1}{4}$	=	$\frac{5}{8}$		
2	+	6	=	8		
$\frac{1}{8}$	+	$\frac{3}{8}$	=	$\frac{1}{2}$		

hrs. & days

21	−	6	=	15		
$\frac{7}{8}$	−	$\frac{1}{4}$	=	$\frac{5}{8}$		
3	+	9	=	12		
$\frac{1}{8}$	+	$\frac{3}{8}$	=	$\frac{1}{2}$		

etc.

lbs. & oz.

4	+	8	=	12		
$\frac{1}{4}$	+	$\frac{1}{2}$	=	$\frac{3}{4}$		

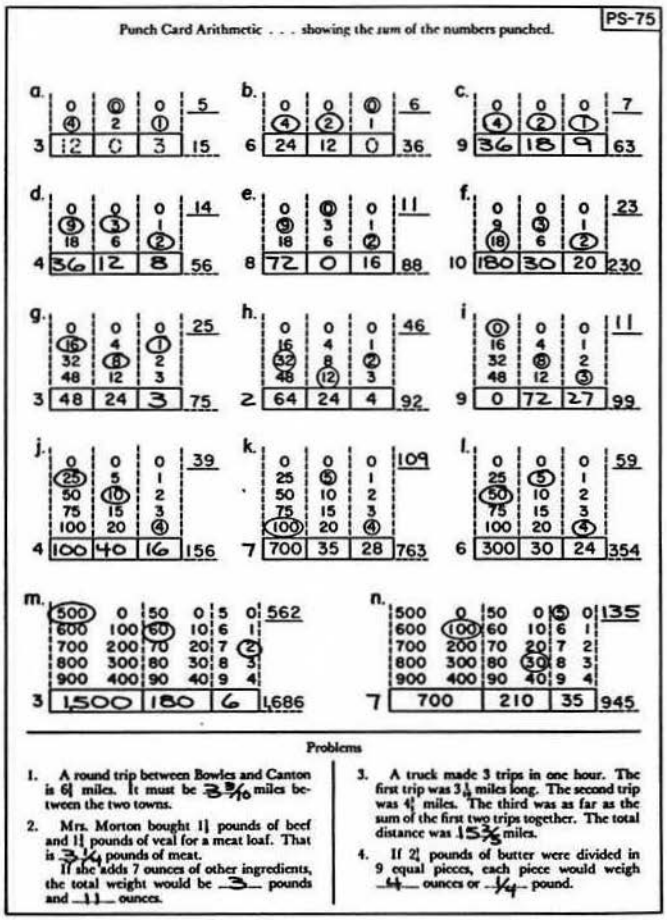
hrs. & days

6	+	12	=	18		
$\frac{1}{4}$	+	$\frac{1}{2}$	=	$\frac{3}{4}$		

gal. & qts.

1	+	2	=	3		
$\frac{1}{4}$	+	$\frac{1}{2}$	=	$\frac{3}{4}$		

Practice sheet 75 contains additional activities like those on practice sheet 74.



PS-75

Punch Card Arithmetic . . . showing the *sum* of the numbers punched.

a. 3 | 12 | 0 | 3 | 15

b. 6 | 24 | 12 | 0 | 36

c. 9 | 36 | 18 | 9 | 63

d. 4 | 36 | 12 | 8 | 56

e. 8 | 72 | 0 | 16 | 88

f. 10 | 180 | 30 | 20 | 230

g. 3 | 48 | 24 | 3 | 75

h. 2 | 64 | 24 | 4 | 92

i. 9 | 0 | 72 | 27 | 99

j. 4 | 100 | 40 | 16 | 156

k. 7 | 700 | 35 | 28 | 763

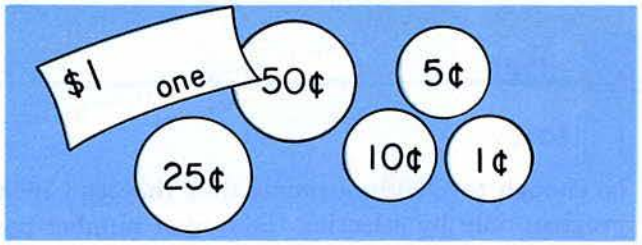
l. 6 | 300 | 30 | 24 | 354

m. 3 | 1,500 | 180 | 6 | 1,686

n. 7 | 700 | 210 | 35 | 945

Problems

1. A round trip between Bowles and Canton is 6⅘ miles. It must be 3 3/10 miles between the two towns.
2. Mrs. Morton bought 1½ pounds of beef and 1¼ pounds of veal for a meat loaf. That is 3 ¼ pounds of meat. If she adds 7 ounces of other ingredients, the total weight would be 3 pounds and 11 ounces.
3. A truck made 3 trips in one hour. The first trip was 3¼ miles long. The second trip was 4½ miles. The third was as far as the sum of the first two trips together. The total distance was 15 ¾ miles.
4. If 2¼ pounds of butter were divided in 9 equal pieces, each piece would weigh 4 ounces or ¼ pound.



	1.	2.	3.
cents	100	50	25
dollars	1	$\frac{1}{2}$	$\frac{1}{4}$

	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
cents	75	10	5	1	2	4	3	27	40	19
dollars	$\frac{3}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{1}{25}$	$\frac{3}{100}$	$\frac{27}{100}$	$\frac{2}{5}$	$\frac{19}{100}$

14. $50 + 75 = 125$

$\frac{1}{2} + \frac{3}{4} = 1\frac{1}{4}$

15. $17 + 33 = 50$

$\frac{17}{100} + \frac{33}{100} = \frac{1}{2}$

16. $10 + 15 + 25 = 50$

$\frac{1}{10} + \frac{3}{20} + \frac{1}{4} = \frac{1}{2}$

17. $40 + 25 = 65$

$\frac{2}{5} + \frac{1}{4} = \frac{13}{20}$

18. $4 + 35 = 39$

$\frac{1}{25} + \frac{7}{20} = \frac{39}{100}$

19. $30 + 20 + 25 = 75$

$\frac{3}{10} + \frac{1}{5} + \frac{1}{4} = \frac{3}{4}$

20. $150 + 210 = 360$

$1\frac{1}{2} + 2\frac{1}{10} = 3\frac{3}{5}$

21. $25 + 20 = 45$

$\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$

22. $30 + 30 + 30 = 90$

$\frac{3}{10} + \frac{3}{10} + \frac{3}{10} = \frac{9}{10}$

23. $2\frac{1}{4} - 1\frac{1}{2} = \frac{3}{4}$

24. $\frac{7}{10} - \frac{1}{5} = \frac{1}{2}$

25. $\frac{1}{20} + \frac{1}{10} + \frac{1}{5} = \frac{7}{20}$

26. $\frac{3}{10} + \frac{9}{10} = 1\frac{1}{5}$

27. $\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$

28. $\frac{1}{4} + \frac{1}{2} + \frac{3}{4} = 1\frac{1}{2}$

29. $\frac{33}{100} + \frac{67}{100} = 1$

30. $\frac{17}{100} - \frac{7}{100} = \frac{1}{10}$

31. $\frac{2}{5} + \frac{3}{5} + \frac{19}{100} = 1\frac{19}{100}$

32. $\frac{3}{25} + \frac{2}{25} = \frac{1}{5}$

33. $\frac{7}{20} - \frac{1}{10} = \frac{1}{4}$

34. $\frac{7}{100} + \frac{11}{100} + \frac{7}{100} = \frac{1}{4}$

Tasks: *In exercises 4-13 pupils complete the Iso Cards. In exercises 14-22 pupils complete the Iso Cards to make true sentences. In exercises 23-34 they complete the bottom sections of Iso Cards to make true sentences.*

Purpose: To develop computational skills for adding and subtracting fractions.

Unifying Ideas: Addition and subtraction.

The Lesson: This set of Ben's Iso Cards has much about it that is familiar. Pupils know that 25¢ is a quarter of a dollar; that 50¢ is half a dollar; that 75¢ is three-quarters of a dollar. They may know that 10¢ is a tenth of a dollar, that a nickel is a twentieth of a dollar, and that a penny is a hundredth of a dollar. However, these last expressions may need some discussion.

Suppose that Ben had 100 pennies. In how many different ways could they be grouped into piles with the same number in each? Start with the smallest number of piles and work up to the largest number that could be used.

number of piles	2	4	5	10	20	25	50	100
pennies in each	50	25	20	10	5	4	2	1

Ben makes some cards to show this:

cents	50	25	20	10	5	4	2	1
dollars	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{50}$	$\frac{1}{100}$

For all other numbers of cents, he uses the fewest cards of one kind that add up to that amount:

3¢ --- $\frac{3}{100}$ 11¢ --- $\frac{11}{100}$ 17¢ --- $\frac{17}{100}$

6¢ --- $\frac{3}{50}$ 12¢ --- $\frac{3}{25}$ 18¢ --- $\frac{9}{50}$

7¢ --- $\frac{7}{100}$ 13¢ --- $\frac{13}{100}$ 19¢ --- $\frac{19}{100}$

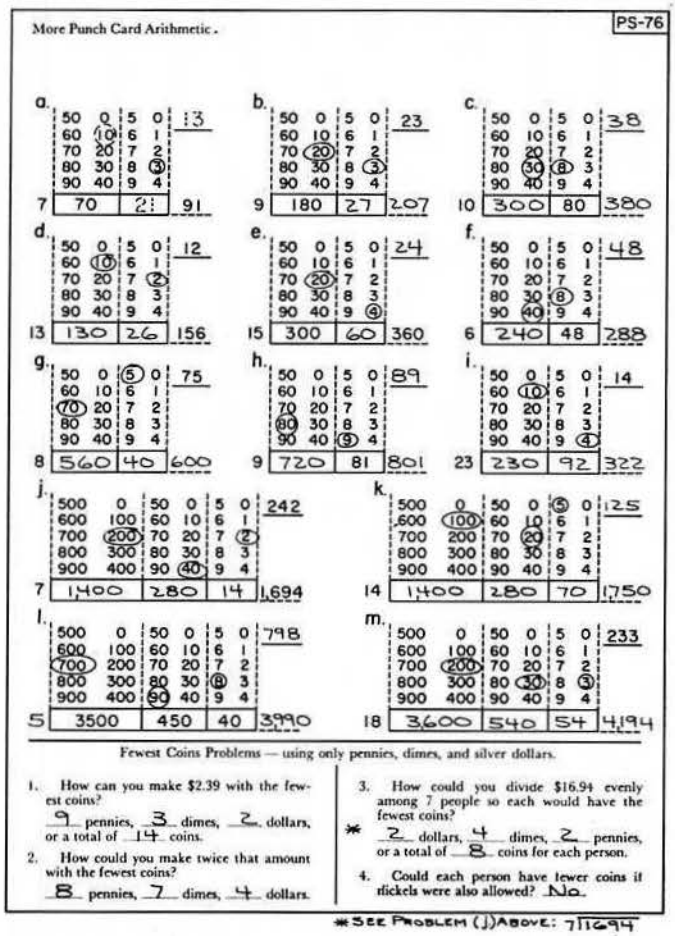
8¢ --- $\frac{2}{25}$ 14¢ --- $\frac{7}{50}$ 21¢ --- $\frac{21}{100}$

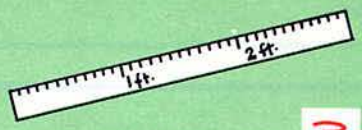
9¢ --- $\frac{9}{100}$ 16¢ --- $\frac{4}{25}$ 22¢ --- $\frac{11}{100}$ etc.

From this point on, one group of pupils might continue with 23¢ through 50¢, another with 51¢ through 75¢, and another with 76¢ through 99¢.

While this may require time and considerable discussion, it is important work that will develop a better understanding of fractions. It is particularly important that every part of the problem can be discussed in this familiar context of coins. The main purpose of this entire development, of course, is to keep emphasizing the relationships between fractions and things.

On practice sheet 76, we have settled on rather familiar lists of numbers to punch out. They are: numbers less than ten, zero and the first 9 whole-number multiples of 10, zero and the first 9 whole-number multiples of 100. As pupils solve each problem, they go through precisely the same steps in the same order as in the usual division algorithm. In the first nine examples, the pupils find the largest whole-number multiple of 10 which, multiplied by the divisor, is less than the dividend. This, of course, is what "finding the trial divisor" is all about. In the last four the trial divisor is the largest whole-number multiple of 100.


More Punch Card Arithmetic.
PS-76
a. 13 7 70 21 91
b. 23 9 180 27 207
c. 38 10 300 80 380
d. 12 13 130 26 156
e. 24 15 300 60 360
f. 48 6 240 48 288
g. 75 8 560 40 600
h. 89 9 720 81 801
i. 14 23 230 92 322
j. 242 7 1400 280 14 1,694
k. 125 14 1400 280 70 1750
l. 798 5 3500 450 40 3990
m. 233 18 3600 540 54 4,194
Fewest Coins Problems — using only pennies, dimes, and silver dollars.
1. How can you make $2.39 with the fewest coins?
9 pennies, 3 dimes, 2 dollars, or a total of 14 coins.
2. How could you make twice that amount with the fewest coins?
8 pennies, 7 dimes, 4 dollars.
3. How could you divide $16.94 evenly among 7 people so each would have the fewest coins?
* 2 dollars, 4 dimes, 2 pennies, or a total of 8 coins for each person.
4. Could each person have fewer coins if nickels were also allowed? No.
*SEE PROBLEM (j) ABOVE: 7|16|94



In 1 yard there are 3 feet.

	1.	2.	3.	4.	5.	6.	7.
feet	3	6	1	2	9	4	5
yards	1	2	$\frac{1}{3}$	$\frac{2}{3}$	3	$1\frac{1}{3}$	$1\frac{2}{3}$

8. $1 + 2 = 3$

$\frac{1}{3} + \frac{2}{3} = 1$

9. $2 + 2 = 4$

$\frac{2}{3} + \frac{2}{3} = 1\frac{1}{3}$

10. $4 + 4 + 4 = 12$

$1\frac{1}{3} + 1\frac{1}{3} + 1\frac{1}{3} = 4$

11. $2 - 1 = 1$

$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

12. $7 - 1 = 6$

$2\frac{1}{3} - \frac{1}{3} = 2$

13. $7 + 11 + 2 = 20$

$2\frac{1}{3} + 3\frac{2}{3} + \frac{2}{3} = 6\frac{2}{3}$

14. $1\frac{1}{3} - \frac{2}{3} = \frac{2}{3}$

15. $3\frac{1}{3} + 7\frac{2}{3} = 11$

16. $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$

17. $4 - 1\frac{1}{3} = 2\frac{2}{3}$

18. $4\frac{1}{3} - 1\frac{2}{3} = 2\frac{2}{3}$

19. $1\frac{1}{3} + 2\frac{2}{3} + 7\frac{1}{3} = 11\frac{1}{3}$

There are 7 days in a week.

	20.	21.	22.	23.	24.	25.	26.
days	7	1	2	3	4	6	9
weeks	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$1\frac{2}{7}$

27. $4 + 2 = 6$

$\frac{4}{7} + \frac{2}{7} = \frac{6}{7}$

28. $13 - 10 = 3$

$1\frac{6}{7} - 1\frac{3}{7} = \frac{3}{7}$

29. $1 + 2 + 3 = 6$

$\frac{1}{7} + \frac{2}{7} + \frac{3}{7} = \frac{6}{7}$

30. $7 - 3 = 4$

$1 - \frac{3}{7} = \frac{4}{7}$

31. $14 - 5 = 9$

$2 - \frac{5}{7} = 1\frac{2}{7}$

32. $3 + 3 + 3 = 9$

$\frac{3}{7} + \frac{3}{7} + \frac{3}{7} = 1\frac{2}{7}$

33. $\frac{6}{7} + 1\frac{1}{7} = 2$

34. $5 - \frac{2}{7} = 4\frac{5}{7}$

35. $1\frac{2}{7} + 2\frac{6}{7} + 1\frac{4}{7} = 5\frac{5}{7}$

36. $\frac{4}{7} + 1\frac{3}{7} = 2$

37. $\frac{5}{7} + \frac{5}{7} = 1\frac{3}{7}$

38. $\frac{5}{7} + \frac{5}{7} + \frac{5}{7} = 2\frac{1}{7}$

77 Yards and feet (thirds) and weeks and days (sevenths)

Tasks: In exercises 1-7 and 20-26 pupils complete the Iso Cards. In exercises 8-12 and 27-32 pupils complete the Iso Cards to make true sentences. In exercises 14-19 and 33-38 they complete the bottom sections of Iso Cards to make true sentences.

Purpose: To develop computational skills for adding and subtracting fractions.

Unifying Ideas: Addition and subtraction.

The Lesson: Here are two sets of Ben's Iso Cards that are particularly easy to work with. In each set, only one denominator occurs, because groups of 3 or 7 cannot be divided except into 3 or 7 groups of 1. Because of this simplicity, considerable practice is provided with measurements that are larger than the basic units—yards and weeks.

Practice sheet 77 provides further experiences with multiples of 1, 10, and 100. Consider example *f*. It asks: $552 \div 8 = ?$ We write this question differently to introduce the usual method of computing:

$$8\overline{)552}$$

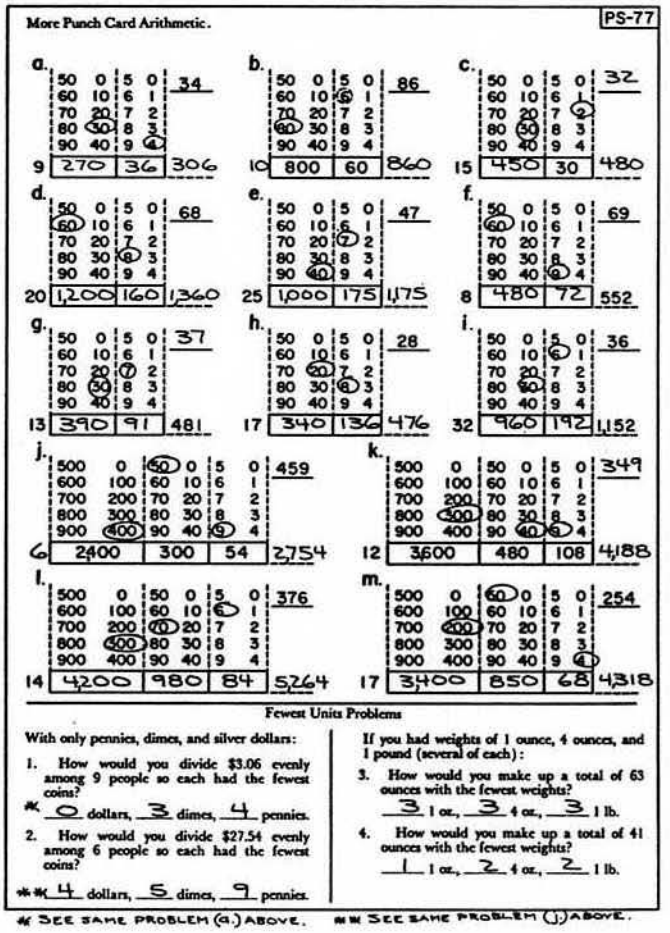
The first step is to find the largest multiple of 10 which, when multiplied by 8, is less than 552. On the practice sheet, that is the first task pupils must do. If they try 70 first, they will find that $70 \times 8 = 560$, which is more than 552. If they try 50 first, they will note that $50 \times 8 = 400$. This leaves $552 - 400$, or 152, to be expressed by multiplying 8 and some number less than 10. This is impossible, so we must go back and try a multiple of 10 that is larger than 50. We find that $60 \times 8 = 480$. This leaves us only 72 to take care of, which is 8×9.

While these are the steps and lines of argument behind the normal division algorithm, they must be performed here without benefit of rules that often obscure the role of place value. Pupils clearly see just what they are doing each step of the way.

In working out these problems, a few trials will be enough to convince pupils that they can make progress only by selecting the largest number possible from each list as they work from left to right. This left-to-right algorithm is a reverse of the right-to-left strategy in multiplication.

This exploration of division will reveal any weakness that pupils have in multiplication by multiples of 10, 100, and, later, 1,000.

Varying the kind of information given from example to example requires pupils to think as they compute—to think in ways that develop a basic understanding of division.


More Punch Card Arithmetic.
PS-77
a. 34 9 270 36 306
b. 86 10 800 60 860
c. 32 15 450 30 480
d. 68 20 1,200 160 1,360
e. 47 25 1,000 175 1,175
f. 69 8 480 72 552
g. 37 13 390 91 481
h. 28 17 340 136 476
i. 36 32 960 192 1,152
j. 459 6 2400 300 54 2,754
k. 349 12 3600 480 108 4,188
l. 376 14 4200 980 84 5,264
m. 254 17 3,400 850 68 4,318
Fewest Units Problems
With only pennies, dimes, and silver dollars:
1. How would you divide $3.06 evenly among 9 people so each had the fewest coins?
* 0 dollars, 3 dimes, 4 pennies.
2. How would you divide $27.54 evenly among 6 people so each had the fewest coins?
** 4 dollars, 5 dimes, 9 pennies.
If you had weights of 1 ounce, 4 ounces, and 1 pound (several of each):
3. How would you make up a total of 63 ounces with the fewest weights?
3 1 oz., 3 4 oz., 3 1 lb.
4. How would you make up a total of 41 ounces with the fewest weights?
1 1 oz., 2 4 oz., 2 1 lb.
* SEE SAME PROBLEM (a.) ABOVE.
** SEE SAME PROBLEM (j.) ABOVE.

Here are cards from three decks of Iso Cards.
Please fill in the incomplete cards below.

	1.	2.	3.	4.	5.	6.	7.	8.	9.
inches	6	12	3	4	1	8	2	9	16
feet	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{3}{4}$	$1\frac{1}{3}$

	10.	11.	12.	13.	14.	15.	16.	17.	18.
tenths of miles	5	10	3	4	1	2	8	15	20
miles	$\frac{1}{2}$	1	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{4}{5}$	$1\frac{1}{2}$	2

	19.	20.	21.	22.	23.	24.	25.	26.	27.
ounces	8	16	3	4	1	12	2	10	22
pounds	$\frac{1}{2}$	1	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{5}{8}$	$1\frac{3}{8}$

The three decks of Iso Cards shown above may give clues
that could help you to complete the sentences shown below.

* 28.

5	+	5	=	10
$\frac{1}{2}$	+	$\frac{1}{2}$	=	1

29.

5	−	2	=	3
$\frac{1}{2}$	−	$\frac{1}{5}$	=	$\frac{3}{10}$

30.

2	+	4	+	3	=	9
$\frac{1}{6}$	+	$\frac{1}{3}$	+	$\frac{1}{4}$	=	$\frac{3}{4}$

* 31.

6	+	4	=	10
$\frac{3}{8}$	+	$\frac{1}{4}$	=	$\frac{5}{8}$

32.

6	+	4	=	10
$\frac{1}{2}$	+	$\frac{1}{3}$	=	$\frac{5}{6}$

33.

15	+	2	+	3	=	20
$1\frac{1}{2}$	+	$\frac{1}{5}$	+	$\frac{3}{10}$	=	2

* 34.

12	−	2	=	10
$\frac{3}{4}$	−	$\frac{1}{8}$	=	$\frac{5}{8}$

35.

7	−	4	=	3
$\frac{7}{10}$	−	$\frac{2}{5}$	=	$\frac{3}{10}$

36.

16	+	26	+	48	=	90
$1\frac{1}{3}$	+	$2\frac{1}{6}$	+	4	=	$7\frac{1}{2}$

* 37.

15	+	21	=	36
$1\frac{1}{4}$	+	$1\frac{3}{4}$	=	3

38.

14	−	5	=	9
$1\frac{2}{5}$	−	$\frac{1}{2}$	=	$\frac{9}{10}$

39.

160	+	56	+	30	=	246
10	+	$3\frac{1}{2}$	+	$1\frac{7}{8}$	=	$15\frac{3}{8}$

40.

$\frac{7}{8}$	−	$\frac{3}{16}$	=	$\frac{11}{16}$

41.

4	+	$\frac{5}{8}$	=	$4\frac{5}{8}$

42.

$1\frac{2}{5}$	+	$1\frac{7}{10}$	+	$1\frac{1}{2}$	=	$4\frac{3}{5}$

* OTHER POSSIBLE WAYS OF FILLING IN THE BLUE BLOCKS.

Tasks: *In exercises 1-27 pupils complete the Iso Cards. In exercises 28-39 pupils complete the Iso Cards to make true sentences. In exercises 40-42 they complete the bottom sections of Iso Cards to make true sentences.*

Purpose: To perfect computational skills for adding and subtracting fractions.

Unifying Ideas: Addition and subtraction.

The Lesson: This pupil page begins with samples from three specified decks of Ben's Iso Cards. In the examples of Iso Card arithmetic that follow, no deck is specified. Which deck is used in exercise 28? In this case, any one of the decks will do. The result doesn't depend on the deck:

ft. (in.)				mi. (10th of mi.)				lbs. (oz.)			
6	+	6	= 12	5	+	5	= 10	8	+	8	= 16
$\frac{1}{2}$	+	$\frac{1}{2}$	= ⋮	$\frac{1}{2}$	+	$\frac{1}{2}$	= ⋮	$\frac{1}{2}$	+	$\frac{1}{2}$	= ⋮

Each of the other exercises has clues that indicate a particular deck—except exercise 37, which could have come from either of two decks:

ft. (in.)				lbs. (oz.)			
15	+	21	= 36	20	+	28	= 48
$1\frac{1}{4}$	+	$1\frac{3}{4}$	= 3	$1\frac{1}{4}$	+	$1\frac{3}{4}$	= 3

Have your pupils make up two kinds of examples: (a) those that can be worked out with only one of the decks, and (b) those that can be worked out with at least two of the decks. This will contribute to their growing awareness of the more general idea of fractions.

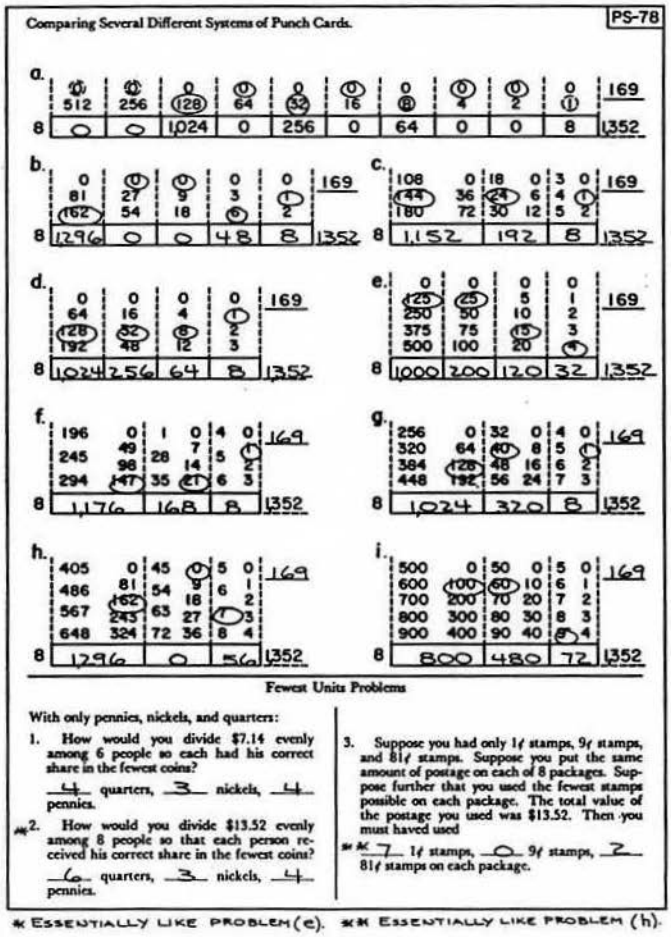
Practice sheet 78 is an overview of division in *Punch Card Arithmetic.* A single example, 1352 ÷ 8, is solved using nine different lists of numbers—with the same result. Some of the lists involve more difficult computations than others. The simplest of all is the final list that contains multiples of 1, 10, and 100. (The list in exercise *e* has some advantages with this particular divisor.)

Do you notice the general pattern in all the lists? Consider several of the second rows:

. . . 8, 4, 2, 1
. . . 27, 9, 3, 1
. . . 64, 16, 4, 1
. . . 125, 25, 5, 1

If you know the number which follows 1 in the sequence (from right to left), that's all you need to know. If it is 4 then each term in the sequence is 4 times the number to the right and ¼ of the number to the left. Our decimal system of numeration follows a similar pattern:

. . . 1,000, 100, 10, 1


Comparing Several Different Systems of Punch Cards.
PS-78
Fewest Units Problems
With only pennies, nickels, and quarters:
1. How would you divide $7.14 evenly among 6 people so each had his correct share in the fewest coins?
4 quarters, 3 nickels, 4 pennies.
*2. How would you divide $13.52 evenly among 8 people so that each person received his correct share in the fewest coins?
6 quarters, 3 nickels, 4 pennies.
3. Suppose you had only 1¢ stamps, 9¢ stamps, and 81¢ stamps. Suppose you put the same amount of postage on each of 8 packages. Suppose further that you used the fewest stamps possible on each package. The total value of the postage you used was $13.52. Then you must haved used
*7 1¢ stamps, 0 9¢ stamps, 2 81¢ stamps on each package.
* Essentially like problem (e). ** Essentially like problem (h).

Here are a few cards from several decks of Iso Cards.
Please fill in the incomplete cards in each deck.

	1.	2.	3.	4.	5.	6.	7.	8.	9.
eggs	6	3	4	2	9	8	1	16	22
dozens of eggs	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{12}$	$1\frac{1}{3}$	$1\frac{5}{6}$

	10.	11.	12.	13.	14.	15.	16.	17.	18.
days	7	1	2	4	5	6	3	9	13
weeks	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{3}{7}$	$1\frac{2}{7}$	$1\frac{6}{7}$

	19.	20.	21.	22.	23.	24.	25.	26.	27.
cents	25	50	75	20	40	10	30	5	15
dollars	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

The three decks of Iso Cards shown above may give clues
that could help you to complete the sentences shown below.

28. $\frac{4}{7} + \frac{2}{7} = \frac{6}{7}$

29. $\frac{1}{2} + \frac{3}{4} = 1\frac{1}{4}$

30. $\frac{1}{10} + \frac{3}{20} + \frac{1}{2} = \frac{3}{4}$

31. $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

32. $1\frac{1}{4} - \frac{1}{20} = 1\frac{1}{5}$

33. $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} = 2\frac{1}{4}$

34. $2 - \frac{3}{7} = 1\frac{4}{7}$

35. $\frac{1}{5} + \frac{1}{4} = \frac{9}{20}$

36. $\frac{5}{6} + \frac{5}{6} + \frac{5}{6} = 2\frac{1}{2}$

37. $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

38. $\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

39. $1\frac{4}{5} + 1\frac{4}{5} + 1\frac{4}{5} = 5\frac{2}{5}$

40. $2\frac{7}{10} + \frac{2}{5} = 3\frac{1}{10}$

41. $3\frac{2}{7} - 1\frac{4}{7} = 1\frac{5}{7}$

42. $2\frac{3}{7} + 2\frac{3}{7} + 2\frac{3}{7} = 7\frac{2}{7}$

43. $\frac{5}{6} - \frac{1}{3} = \frac{1}{2}$

44. $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

45. $1\frac{4}{5} + 1\frac{3}{10} + 4\frac{1}{2} = 7\frac{3}{5}$

Notes on Pupil Page 79

Tasks: In exercises 1-27 pupils complete the Iso Cards. In exercises 28-45 they complete the bottom sections of Iso Cards to make true sentences.

Purpose: To perfect computational skills for adding and subtracting fractions.

Unifying Ideas: Addition and subtraction.

The Lesson: This pupil page begins with sample cards from three different decks of Ben's Iso Cards. While the first is labeled "eggs and dozens of eggs," it could just as well be the inches-feet deck. The days-week deck is one in which all fractions have the same denominator. The third is the most complicated and yet the most familiar deck of all.

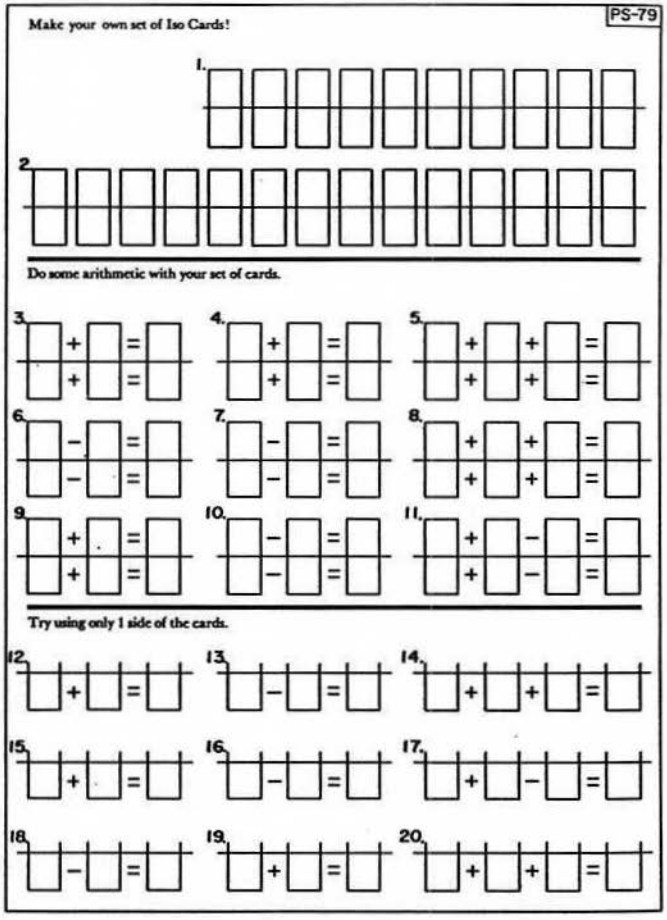
In what other situations might each of these decks be useful? The first could be used in talking about inches and feet. It might also be used to describe the situation if Mary wanted to save $12.00. When she had saved the first dollar, she would have one-twelfth of the job done; $6.00 would be the half-way point. She would be eleven-twelfths done with her task when she had saved $5.00 beyond the half-way mark. If at that point she received a birthday gift of $7.00, giving her a total of $18.00, we could say her original goal has been reached one and one-half times.

The second deck on this pupil page could be used to describe single caramels and 7-caramel packages at the Caramel Company—3 caramels and three-sevenths of a package are the same thing, as are 12 caramels and one and five-twelfths packages.

The third set of cards can be used to help talk about a trip to grandmother's house, 100 miles away. Fifty miles would be half-way; 80 miles would be four-fifths of the way. When only a mile is left to go, the trip would be ninety-nine-hundredths completed. On the way back, after 25 miles had been covered, someone might say: We've gone as far as one and a quarter trips to grandmother's house. Or, more probably, Our round trip is five-eighths completed.

Have pupils suggest other uses for these decks.

Practice sheet 79 is another blank sheet on which children can practice making up decks of their own and doing Iso Card arithmetic.



I. Draw a loop around the pair that makes the sentence true.

Sample

□ x 6 = (4 x ○) + 2

4, 6 — 2, 3 — 5, 7 — 0, 1

A.

4 x 7 = (△ + □) − 9

28, 10 — 15, 22 — 19, 0 — 6, 6

B.

(4 x ○) + 4 = 5 x △

9, 8 — 16, 8 — 5, 4 — 7, 9

C.

△ x ○ = (5 x 9) − 3

8, 6 — 2, 15 — 6, 7 — 13, 3

D.

□ + □ = ○ x ○

1, 1 — 4, 4 — 0, 7 — 2, 2

E.

△ + □ = (7 x 8) + 4

78, 4 — 23, 37 — 26, 30 — 15, 4

F.

2 x (□ + △) = 5 x 6

6, 9 — 10, 12 — 2, 30 — 15, 7

G.

△ x ○ = 27 x 2

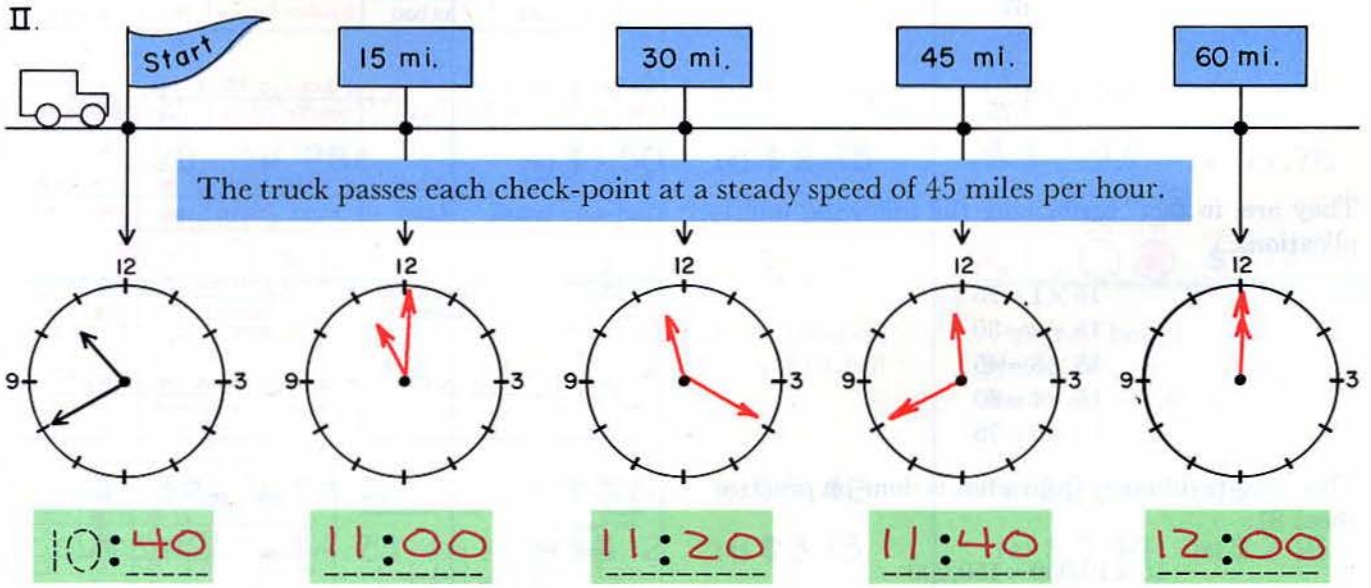
13, 0 — 9, 6 — 3, 9 — 6, 18

H.

(○ x 7) − 6 = (7 x □) − 13

42, 91 — 1, 6 — 8, 9 — 7, 10

II.



Show the hands of the clock and the time shown by the clock at each of the check-point marks.

80-81 Following Directions: Achievement Tests

Tasks: In exercises A-H pupils draw a loop around the pair of numbers that makes the sentence a true statement. In exercise II pupils draw the hands of each clock and record the time shown on the clock at each of the checkpoints.

Purpose: To perfect computational skills with whole numbers by exploring equations. To explore concepts of uniform motion.

Unifying Ideas: Sets; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: This is the first of two pupil pages you may use as a checkup. The first nine exercises may seem a bit strange to some pupils, and the idea ought to be discussed before independent work is undertaken.

We suggest you call on "gremlins"—strange characters who delight in erasing parts of statements in arithmetic—to help out. Ask the class for several true statements to write on the board. Then take them up one at a time:

$$(3 \times 7) + 4 = 30 - 5$$

The gremlins came along, erased two of the numbers in this sentence, and left frames to indicate where they had made erasures:

$$(3 \times \square) + 4 = 30 - \triangle$$

If you know nothing more, you wouldn't be sure what they had erased. However, they left a note saying that the pair they erased was one of these:

$\boxed{4}, \triangle 1$; $\boxed{10}, \triangle 4$; $\boxed{7}, \triangle 5$; $\boxed{6}, \triangle 12$

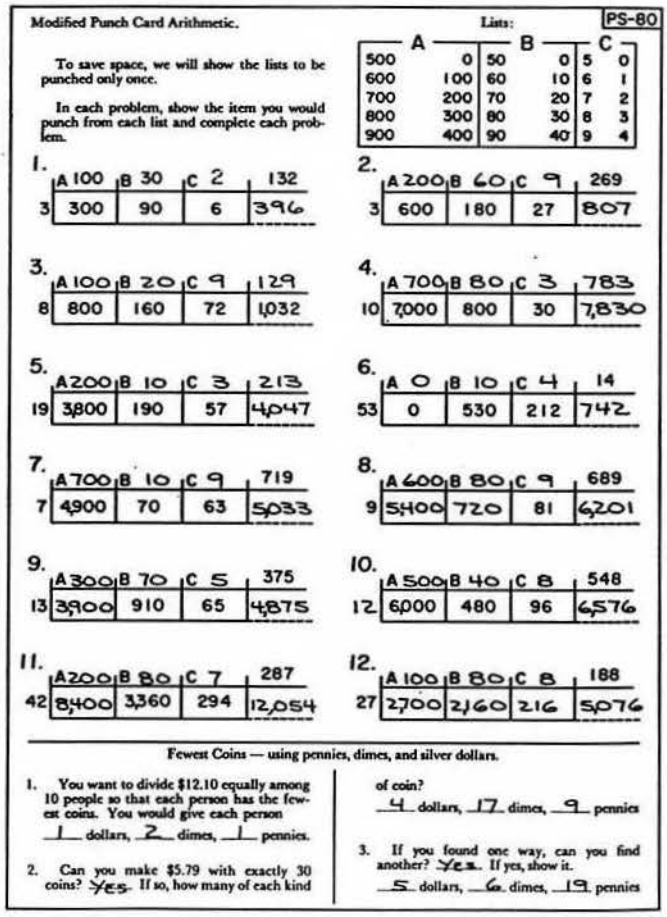
Now, are you sure what the sentence said before the erasures? Try the numbers one pair at a time.

$(3 \times 4) + 4 = 30 - 1$ _____ false
$(3 \times 10) + 4 = 30 - 4$ _____ false
$(3 \times 7) + 4 = 30 - 5$ _____ true
$(3 \times 6) + 4 = 30 - 12$ _____ false

It must have been the third pair. With several such examples, the class ought to have no trouble doing this pupil page independently.

Exercise II, with five parts, will check on ability to tell time by the clock. If the method of recording time—such as 10:40—is a problem, discuss it with your pupils before they work alone.

Punch Card Arithmetic is presented in a slightly modified form on practice sheet 80. Rather than repeating the list above each exercise, we have given a single list at the top of the page. Instead of circling one of the numbers in each section, pupils write the numbers in the spaces indicated by A, B, and C. This change is another step toward the usual division algorithm.



Modified Punch Card Arithmetic. PS-80

To save space, we will show the lists to be punched only once.

In each problem, show the item you would punch from each list and complete each problem.

Lists:

A	B	C
500	0	50	0	5	0
600	100	60	10	6	1
700	200	70	20	7	2
800	300	80	30	8	3
900	400	90	40	9	4

1.

A 100	B 30	C 2	132
3 300	90	6	396

2.

A 200	B 60	C 9	269
3 600	180	27	807

3.

A 100	B 20	C 9	129
8 800	160	72	1,032

4.

A 700	B 80	C 3	783
10 7,000	800	30	7,830

5.

A 200	B 10	C 3	213
19 3,800	190	57	4,047

6.

A 0	B 10	C 4	14
53 0	530	212	742

7.

A 700	B 10	C 9	719
7 4,900	70	63	5,033

8.

A 600	B 80	C 9	689
9 5,400	720	81	6,201

9.

A 300	B 70	C 5	375
13 3,900	910	65	4,875

10.

A 500	B 40	C 8	548
12 6,000	480	96	6,576

11.

A 200	B 80	C 7	287
42 8,400	3,360	294	12,054

12.

A 100	B 80	C 8	188
27 2,700	2,160	216	5,076

Fewest Coins — using pennies, dimes, and silver dollars.

1. You want to divide $12.10 equally among 10 people so that each person has the fewest coins. You would give each person 1 dollars, 2 dimes, 1 pennies.

2. Can you make $5.79 with exactly 30 coins? Yes. If so, how many of each kind of coin? 4 dollars, 17 dimes, 9 pennies

3. If you found one way, can you find another? Yes. If yes, show it. 5 dollars, 6 dimes, 19 pennies

Following Directions

$7 \times 7 =$ (1) 42 (2) 49 (3) 60 (4) 56

Mark the correct answer. Mark the furthest from the correct answer.

(1) (2) (3) (4) The difference between the answers you marked.

1 A quarter is worth how much more than a dime?

(1) 35¢ (3) 15¢
(2) 41¢ (4) 10¢

(1) (2) (3) (4) 26¢

2 Bill earns 50¢ a week. How much does he earn in 13 weeks?

(1) $6.50 (3) $5.53
(2) $13.50 (4) $12.00

(1) (2) (3) (4) $7.00

3 What is the cost of 2 dozen sheets of cardboard at 4 sheets for 25¢.

(1) $1.00 (3) $1.50
(2) $2.75 (4) $2.00

(1) (2) (3) (4) $1.25

4 Cathy bought 2 cans of peas at 27¢ each and 3 cans of corn at 21¢ each. How much did she spend?

(1) $1.17 (3) $1.07
(2) 96¢ (4) $1.16

(1) (2) (3) (4) 21¢

5 How much change should Alice get back? She bought a book for 79¢ and a game for 90¢. She gave the clerk $2.00.

(1) $1.69 (3) 31¢
(2) $3.69 (4) 19¢

(1) (2) (3) (4) $3.38

6 Harry bought 3 note books. "That's $\frac{1}{4}$ of all they had," he said. How many did the store have before Harry made his purchase?

(1) 7 (3) 4
(2) 16 (4) 12

(1) (2) (3) (4) 8

Sale

Belt	75¢	Scarf	$1.14
Blouse	$2.29	Dress	$3.58

7 A dress costs how much more than a blouse?

(1) $1.31 (3) $5.87
(2) $1.29 (4) 28¢

(1) (2) (3) (4) $4.58

8 Four belts cost:

(1) $3.00 (3) $4.75
(2) $1.50 (4) $2.75

(1) (2) (3) (4) $1.75

9 Altogether, one of each item would cost:

(1) $10.00 (3) $7.55
(2) $9.93 (4) $7.76

(1) (2) (3) (4) $2.24

10 Two blouses and one scarf would cost:

(1) $4.54 (3) $5.72
(2) $3.42 (4) $4.73

(1) (2) (3) (4) $2.30

11 Mary bought 2 blouses. Her change from a $10 bill should be:

(1) $7.71 (3) $5.42
(2) $3.12 (4) $3.13

(1) (2) (3) (4) $2.30

12 Alice bought 2 blouses. Helen bought a scarf. How much more did Alice spend?

(1) $2.17 (3) $4.71
(2) $3.44 (4) $2.16

(1) (2) (3) (4) $1.28

Tasks: Pupils follow the indicated directions to complete each exercise.

Purpose: To assess pupil achievement in applications.

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

The Lesson: Here are more problems that use the multiple-choice format of many standardized tests. The style of question, the vocabulary, and the level of computational ability required is comparable to much that is found in standardized tests at about beginning fifth grade level. The fact that a large number of pupils miss questions like these after four years of arithmetic has led us to find more effective ways of helping pupils learn how to handle computations with increased facility.

Punch Card Division is developed further on practice sheet 81. The list of numbers to be punched is extended to include multiples of 1,000 and 10,000.

When pupils learn the final algorithm they will work the fifth exercise in this way:

$$
\begin{array}{r}
12{,}345 \\
15\overline{)185{,}175} \\
15 \\
\hline
35 \\
30 \\
\hline
51 \\
45 \\
\hline
67 \\
60 \\
\hline
75 \\
75 \\
\hline
0
\end{array}
$$

They are, in fact, performing the following multiplications:

$$
\begin{aligned}
15 \times 1 &= 15 \\
15 \times 2 &= 30 \\
15 \times 3 &= 45 \\
15 \times 4 &= 60 \\
15 \times 5 &= 75
\end{aligned}
$$

This is quite different from what is done on practice sheet 81:

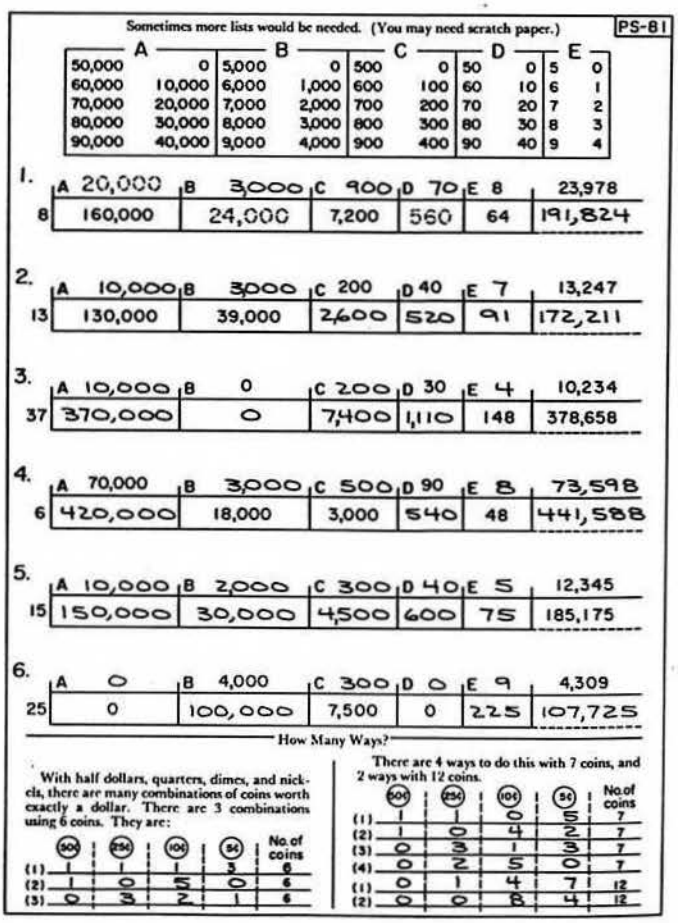
$$
\begin{aligned}
15 \times 10{,}000 &= 150{,}000 \\
15 \times 2{,}000 &= 30{,}000 \\
15 \times 300 &= 4{,}500 \\
15 \times 40 &= 600 \\
15 \times 5 &= 75
\end{aligned}
$$

However, it is only through experience with this expanded form that pupils can develop real understanding of the final algorithm. This expanded form also leads to an undertanding of the extension of ordinary numerals to decimal fractions, to be studied later.

As pupils add the partial quotients:

10,000, 2,000, 300, 40, 5 . . . 12,345

they find it ridiculously simple. This will help them understand an important part of the extremely efficient division algorithm we have all learned. To realize that all the algorithms are carefully worked out plans to break up big problems into little ones and to facilitate the necessary bookkeeping is a big step toward understanding the rationale for the whole subject of elementary school arithmetic.



Sometimes more lists would be needed. (You may need scratch paper.) PS-81

A		B		C		D		E	
50,000	0	5,000	0	500	0	50	0	5	0
60,000	10,000	6,000	1,000	600	100	60	10	6	1
70,000	20,000	7,000	2,000	700	200	70	20	7	2
80,000	30,000	8,000	3,000	800	300	80	30	8	3
90,000	40,000	9,000	4,000	900	400	90	40	9	4

1.

	A	B	C	D	E	
8	160,000	24,000	7,200	560	64	191,824

2.

	A	B	C	D	E	
13	130,000	39,000	2,600	520	91	172,211

3.

	A	B	C	D	E	
37	370,000	0	7,400	1,110	148	378,658

4.

	A	B	C	D	E	
6	420,000	18,000	3,000	540	48	441,588

5.

	A	B	C	D	E	
15	150,000	30,000	4,500	600	75	185,175

6.

	A	B	C	D	E	
25	0	100,000	7,500	0	225	107,725

How Many Ways?

With half dollars, quarters, dimes, and nickels, there are many combinations of coins worth exactly a dollar. There are 3 combinations using 6 coins. They are:

	50¢	25¢	10¢	5¢	No. of coins
(1)	1	1	1	3	6
(2)	1	0	5	0	6
(3)	0	3	2	1	6

There are 4 ways to do this with 7 coins, and 2 ways with 12 coins.

	50¢	25¢	10¢	5¢	No. of coins
(1)	1	1	0	5	7
(2)	1	0	4	2	7
(3)	0	3	1	3	7
(4)	0	2	5	0	7
(1)	0	1	4	7	12
(2)	0	0	8	4	12

Jean Makes a Point

"I get a little tired." Jean objected to always having to think about Ben's things — his eggs and cookies and minutes and miles.

"I don't care what Ben is talking about . . .

> "I know that $\frac{1}{2}$ of anything plus $\frac{1}{4}$ of the same kind of thing equals $\frac{3}{4}$ of whatever you are talking about.

"So, I write:

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

and that is true unless someone wants to talk about special ideas . . . and if they do, they should explain.

"In each of the examples below, you can think of whatever things you like and the results will be the same."

1. $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$

2. $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

3. $\frac{3}{4} + \frac{1}{4} = 1$

4. $\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$

5. $1\frac{1}{10} + 1\frac{2}{5} = 2\frac{1}{2}$

6. $\frac{5}{6} - \frac{1}{3} = \frac{1}{2}$

7. $2\frac{3}{4} + 1\frac{1}{2} = 4\frac{1}{4}$

8. $3\frac{1}{3} - 1\frac{1}{2} = 1\frac{5}{6}$

9. $7\frac{3}{10} + 2\frac{4}{5} = 10\frac{1}{10}$

"Now," Jean continued, "I don't care what set of Ben's cards you think about . . . we all get the same results."

Mrs. Brown put each example on the board and asked for volunteers to give their answers.

"In the second example," Arleen said, "I wrote the answer differently. I was thinking of a pie that was cut into 6 pieces, so I wrote:

$$\frac{1}{3} + \frac{1}{6} = \frac{3}{6}$$

and I think that $\frac{1}{2}$ and $\frac{3}{6}$ mean the same thing. So, I think my answer is correct also."

Frank had a similar difference of opinion about example 5.

"I thought about miles — and tenths of a mile. I wrote:

$$1\frac{1}{10} + 1\frac{2}{5} = 2\frac{5}{10}$$

"I could have changed that to $2\frac{1}{2}$ miles, but people often talk about $2\frac{5}{10}$ miles . . . only they usually write 2.5 miles."

Everyone agreed that the way Arleen and Frank wrote their results made good sense.

Mary put the idea in words when she said

> "There are different ways to talk about the same thing."

And Ben, always ready to go back to his Arithmetic of Things and Numbers, asked to explain one of his ideas.

Tasks: *Pupils complete each sentence to make it a true statement.*

Purpose: To perfect computational skills for adding and subtracting fractions.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

The Lesson: Many pupils will share Jean's objections. They will be a little annoyed at the continual intrusion of some particular thing into the discussion. They will have noticed that if the question

$$\frac{1}{2}+\frac{1}{4}=\text{\_\_\_\_}$$

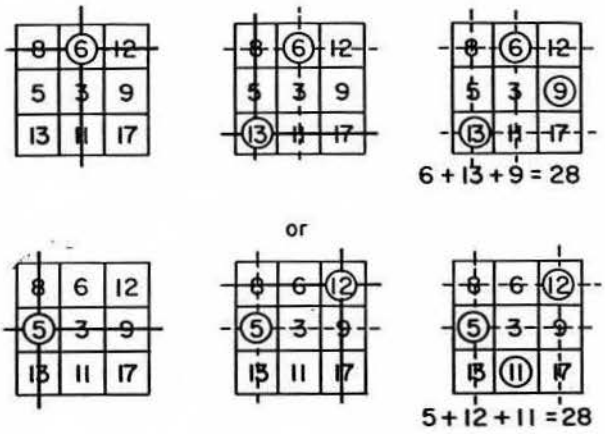
arises in any context whatever, the answer will be the same whether it's dollars, feet, hours, or gallons. Thus, the statement

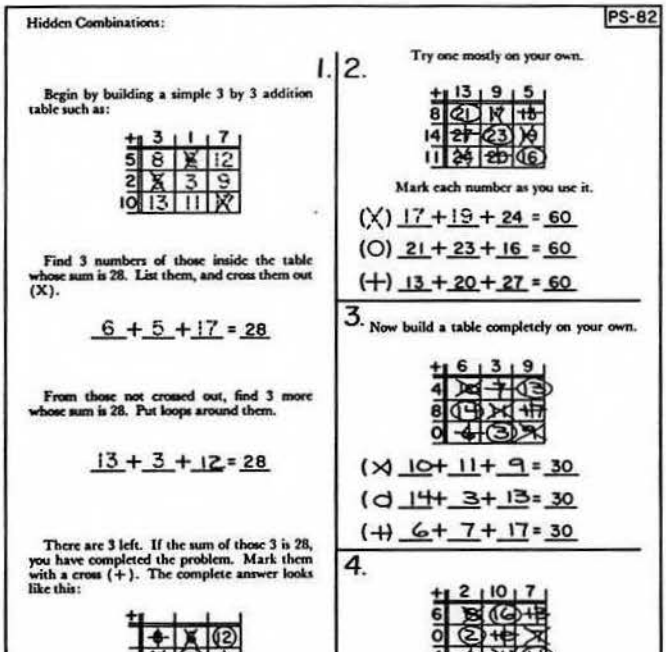
$$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

has a kind of independent existence all its own, divorced from any particular physical situation.

The story of the *Arithmetic of Things and Numbers* was selected by the authors as one way to avoid a mistake that caused lack of understanding in the past—the mistake of abstracting numbers, particularly fractions, too early in a development. A game that follows formal, arbitrary rules—the mathematicians' game—is not a game many pupils will choose if given an alternative. Most pupils would rather work in the world of things, learning to talk sensibly about experiments they perform or imagine performing. They prefer to think about manipulations of things according to the rules of the real world rather than about manipulations of symbols in compliance with formal agreements.

The hidden combinations activity introduced on practice sheet 82 begins as a bit of a puzzle—and, incidentally, involves lots of practice in mental addition. There is a quick way to find these combinations. Select any number, then cross out the row and column it is in. Pick one of the numbers not crossed out. Then cross out the row and column in which it appears. This leaves one number. The sum of the two numbers, plus the one number left over, will be the same, no matter where you start.

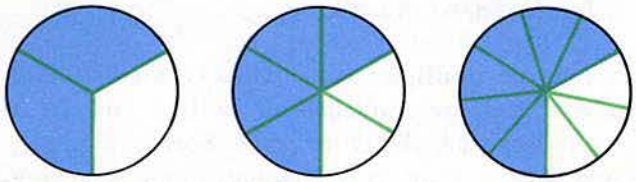



Hidden Combinations:
PS-82
1.
2.
Try one mostly on your own.
Begin by building a simple 3 by 3 addition table such as:
Find 3 numbers of those inside the table whose sum is 28. List them, and cross them out (X).
6 + 5 + 17 = 28
From those not crossed out, find 3 more whose sum is 28. Put loops around them.
13 + 3 + 12 = 28
There are 3 left. If the sum of those 3 is 28, you have completed the problem. Mark them with a cross (+). The complete answer looks like this:
(X) 6 + 5 + 17 = 28
(O) 13 + 3 + 12 = 28
(+) 8 + 9 + 11 = 28
Mark each number as you use it.
(X) 17 + 19 + 24 = 60
(O) 21 + 23 + 16 = 60
(+) 13 + 20 + 27 = 60
3.
Now build a table completely on your own.
(X) 10 + 11 + 9 = 30
(O) 14 + 3 + 13 = 30
(+) 6 + 7 + 17 = 30
4.
(X) 7 + 8 + 14 = 29
(O) 2 + 11 + 16 = 29
(+) 6 + 10 + 13 = 29
What Can You Say?
I have 50¢ in my pocket. What can you say?
The smallest number of coins you can have is ____. The largest number is 50 If you have 2 coins, I know they are both
QUARTERS Could I have exactly 3 coins? No. Could I have exactly 40 coins? No Could I have exactly 11 coins? No

"Remember my

$\overline{3},2$

way of giving directions? Well, I found out that many different directions would lead to the same general result. For example:



$\overline{3},2$ $\overline{6},4$ $\overline{9},6$

"Each of these directions names the same amount of pie.

"Here is another example."

$\overline{2},1$ $\overline{4},2$ $\overline{6},3$ $\overline{8},4$

"Each direction refers to the same length of board."

Jean returned to the discussion. "Again, I think that Ben is too interested in special things — boards and pies. I think that I could say many things about:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$$

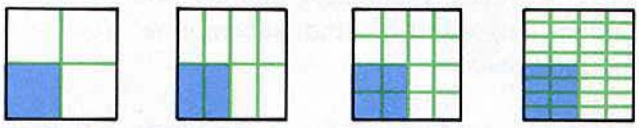
and:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$$

"As Mary said, they are different ways to talk about the same amount of whatever it is you are talking about."

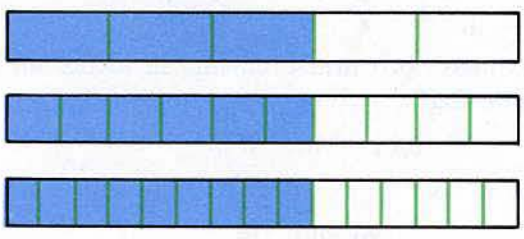
"Are there different ways to talk about fractions?" Al asked.

"I think so," was Ben's quick answer. "Because when you cut something up and talk about some of the pieces, you can always cut it up into smaller pieces and talk about these new pieces." He drew a sketch:



$\overline{4},1$ $\overline{8},2$ $\overline{16},4$ $\overline{32},8$

and



$\overline{5},3$ $\overline{10},6$ $\overline{15},9$

"Ruth and I used to make tables for Mother that looked like this. I'll leave blanks for you to complete.

A.

$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{6}{12}$	$\frac{7}{14}$	$\frac{8}{16}$	—

B.

$\frac{2}{3}$	$\frac{4}{6}$	$\frac{6}{9}$	$\frac{8}{12}$	$\frac{10}{15}$	$\frac{12}{18}$	$\frac{14}{21}$	$\frac{16}{24}$	—

C.

$\frac{1}{4}$	$\frac{2}{8}$	$\frac{3}{12}$	$\frac{4}{16}$	$\frac{5}{20}$	$\frac{6}{24}$	$\frac{7}{28}$	$\frac{8}{32}$	—

D.

$\frac{3}{5}$	$\frac{6}{10}$	$\frac{9}{15}$	$\frac{12}{20}$	$\frac{15}{25}$	$\frac{18}{30}$	$\frac{21}{35}$	$\frac{24}{40}$	—

"And we always started our lists by noting the simplest way of writing about a fraction."

83 Equivalent fractions

Tasks: Pupils read and discuss the story and fill in the blanks in exercises A-D.

Purpose: To explore the concept of classes of equivalent fractions.

Unifying Ideas: Structure; number and counting; numeration; functions and relations.

The Lesson: We use the fact that the denominator specifies some sub-unit and the numerator specifies the number of these sub-units to help pupils understand that there are whole families of equivalent fractions. If we divide a stick into 5 parts we can call each part a fifth. And if we shade 3 of these parts we can describe this shaded section as three-fifths.

$\frac{3}{5}$

To find another way to talk about the same shaded section, we can break each fifth into as many parts as we like, say three, and describe the shaded part as nine-fifteenths:

$\frac{9}{15}$

Now we can divide each of the fifteenths into as many parts as we like—two, for example:

$\frac{18}{30}$

We can repeat this process again, splitting each thirtieth in half:

$\frac{36}{60}$

Someone might note that we now have as many pieces as there are minutes in an hour. So we can say that 36 minutes might be called

$$\frac{36}{60} \text{ or } \frac{18}{30} \text{ or } \frac{9}{15} \text{ or } \frac{3}{5} \text{ of an hour.}$$

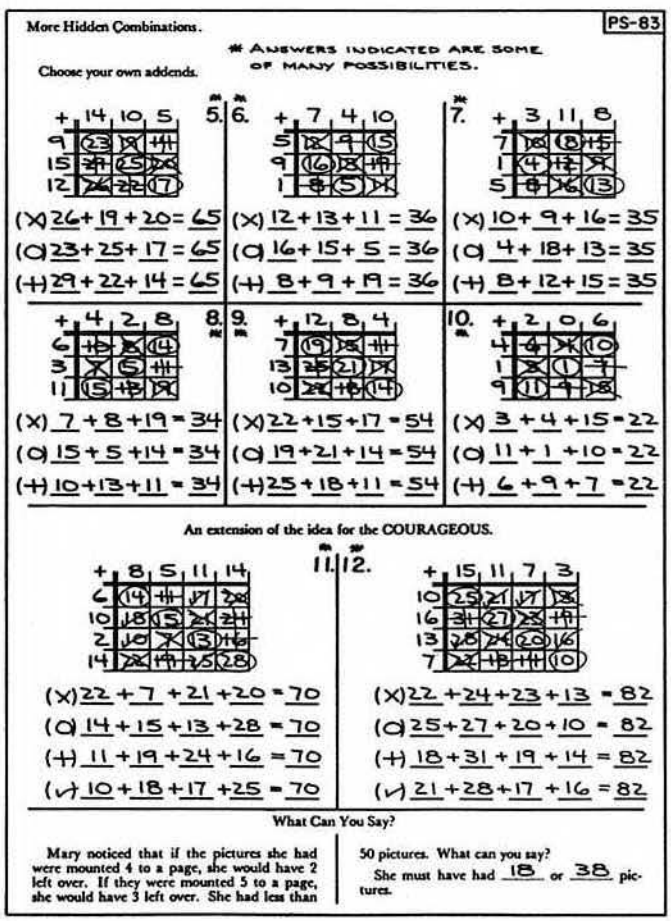
Of course, if we had chosen other divisions as we went along, we would have found:

$$\frac{6}{10}, \frac{12}{20}, \frac{15}{25}, \frac{21}{35}, \frac{24}{40}, \frac{27}{45}, \frac{30}{50}, \frac{33}{55}, \text{ etc.}$$

On pupil page 83 Ben points out that in each of the families there is always one fraction that is the "simplest"—one that has the smallest denominator and the smallest numerator. We may say that it is the fraction that describes the least breaking up and the least counting or uses the largest possible sub-unit. Pupils might well be advised that most test-makers require the final answer to be this "simplest" fraction.

Practice sheet 83 provides more experimentation in hidden combinations. The quick way of finding combinations, suggested for practice sheet 82, can be extended as the tables get larger.

Exercises 11 and 12 provide forms for four-by-four tables. Perhaps the class, a group, or an individual would like to try tables that are five by five, six by six, etc. They will find that larger tables behave much like smaller ones. Along the way, of course, there's lots of practice in mental addition.


More Hidden Combinations.
PS-83
* Answers indicated are some of many possibilities.
Choose your own addends.
5.
6.
7.
8.
9.
10.
An extension of the idea for the COURAGEOUS.
11.
12.
What Can You Say?
Mary noticed that if the pictures she had were mounted 4 to a page, she would have 2 left over. If they were mounted 5 to a page, she would have 3 left over. She had less than 50 pictures. What can you say?
She must have had 18 or 38 pictures.

Ruth added to Ben's explanation.

"When fractions are greater than 1, we used to make a double list, because you can think of them in two different ways. Using our

$\overline{4},5$

way of giving directions, we can sketch the idea this way:

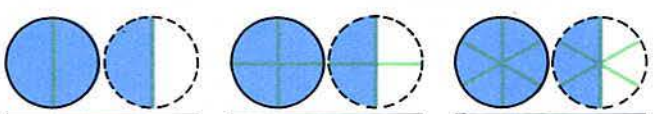
$\overline{4},5$ $\overline{8},10$

"This leads to our double charts:

A.

$1\frac{1}{4}$	$1\frac{2}{8}$	$1\frac{3}{12}$	$1\frac{4}{16}$	$1\frac{5}{20}$	$1\frac{6}{24}$
$\frac{5}{4}$	$\frac{10}{8}$	$\frac{15}{12}$	$\frac{20}{16}$	$\frac{25}{20}$	$\frac{30}{24}$

"Here is another example."



$\overline{2},3$ $\overline{4},6$ $\overline{6},9$

B.

$1\frac{1}{2}$	$1\frac{2}{4}$	$1\frac{3}{6}$	$1\frac{4}{8}$	$1\frac{5}{10}$	$1\frac{6}{12}$
$\frac{3}{2}$	$\frac{6}{4}$	$\frac{9}{6}$	$\frac{12}{8}$	$\frac{15}{10}$	$\frac{18}{12}$

"Here is another without a sketch."

C.

$2\frac{1}{3}$	$2\frac{2}{6}$	$2\frac{3}{9}$	$2\frac{4}{12}$	$2\frac{5}{15}$	$2\frac{6}{18}$
$\frac{7}{3}$	$\frac{14}{6}$	$\frac{21}{9}$	$\frac{28}{12}$	$\frac{35}{15}$	$\frac{42}{18}$

Jack had a question for Ben.

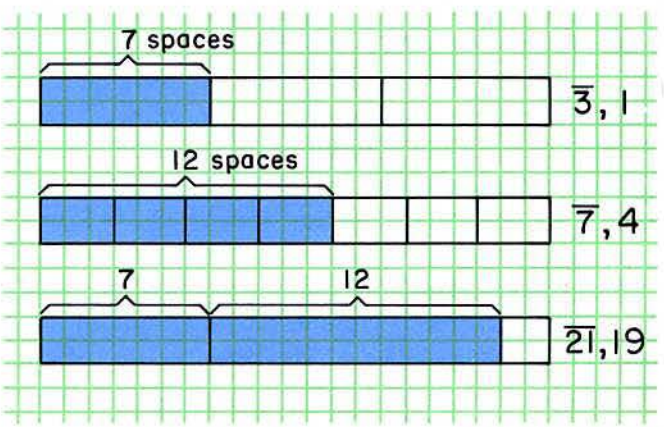
"What do you do, Ben, when you run across a problem such as

$$\frac{1}{3} + \frac{4}{7}$$

and you don't have any set of your cards you can think about? Eggs won't fit . . . minutes . . . days — none of these is usually divided into three parts and also into seven parts. What do you do?"

"Well, I used to work out something using a piece of graph paper. I wanted to draw a picture of a stick that I could easily cut into 3rds and into 7ths. How long should I draw it?

"What I really wanted to do was find a number that could be easily divided by 3 and by 7. Of course, 21 would do. (So would 42.) So I drew my stick 21 spaces long. And you can see my method:



"So, I would conclude that

$$\frac{1}{3} + \frac{4}{7} = \frac{19}{21}.$$

Jean pointed out right away, "Well, I would tell more of your story by writing:

$$\frac{1}{3} + \frac{4}{7} = \frac{7}{21} + \frac{12}{21} = \frac{19}{21}$$

84-85 Introduction of usual algorithm - using common denominators

Tasks: Pupils read and discuss the story about addition of fractions and complete each chart.

Purpose: To explore a rationale for an addition algorithm for fractions.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; functions and relations.

The Lesson: Most everyday problems involving fractions arise in a context that suggests a particular common denominator. Consider:

$$\frac{5}{6}\text{ yd} + \frac{1}{4}\text{ yd} = \_\_\_\_\text{ yd}$$

We can express both measurements in inches and can conclude that

$$30\text{ in} + 9\text{ in} = 39\text{ in}$$

Or, we might say:

1 yd and 3 in

or:

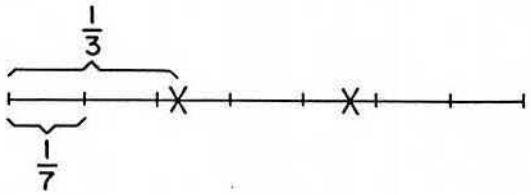
$$1\frac{3}{36}\text{ yd or } 1\frac{1}{12}\text{ yd}$$

However, situations will arise in which pupils are asked to supply the answer to

$$\frac{1}{3} + \frac{4}{7} = ?$$

with no context given to provide clues to the denominator to use.

Ben's solution is a good one. He simply imagines that the problem is about a stick. The stick can be any length he wants. And, since he is going to sketch the stick on graph paper, he selects a length that will make it possible to use the graph-paper rulings to help him divide the stick into thirds and sevenths. How many graph-paper squares long should he make his unit stick? Here's a sketch of his problem:



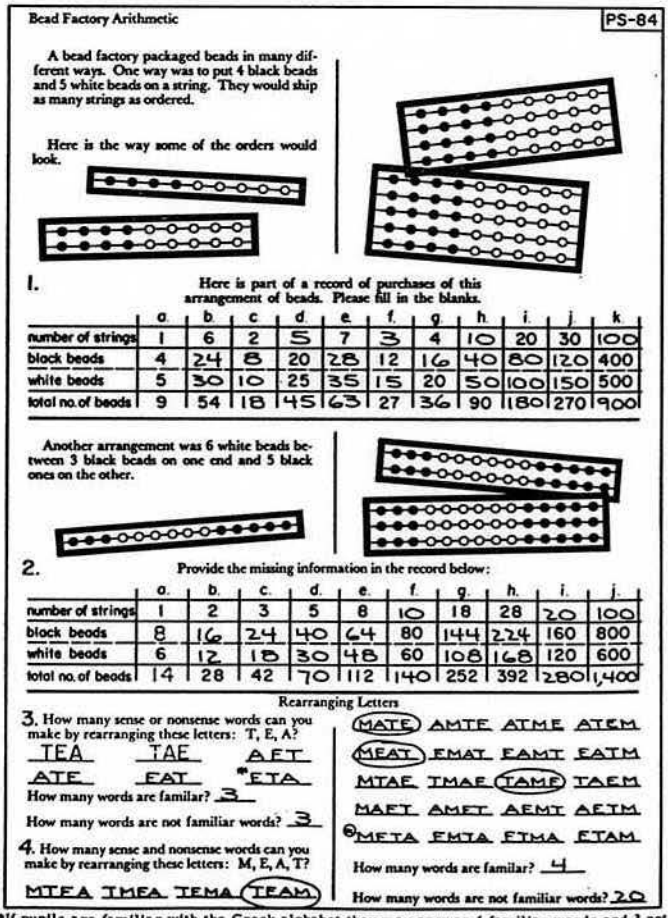
How can he plan ahead so each division point will fall on a graph-paper ruling? Any multiple of 3 would do for ⅓ and any multiple of 7 would do for ⅐. He could make two lists of these multiples:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, etc.

7, 14, 21, 28, 35, 42, 47, etc.

The lists of multiples have already yielded two numbers that are multiples of both 3 and 7. If he wants his stick-sketch to be as short as possible, he will choose to make it 21 squares long, knowing of course, that the larger sketch would lead to the same result. Using 21 sub-units, the problem now becomes one of adding 7 sub-units + 12 sub-units = 19 sub-units. Each sub-unit is, of course, ⅟₂₁ of the stick.

On practice sheet 84 we begin a four-page excursion into Bead Factory Arithmetic. This is a thinly disguised review of the multiplication algorithm.



Bead Factory Arithmetic — PS-84

A bead factory packaged beads in many different ways. One way was to put 4 black beads and 5 white beads on a string. They would ship as many strings as ordered.

Here is the way some of the orders would look.

1. Here is part of a record of purchases of this arrangement of beads. Please fill in the blanks.

	a	b	c	d	e	f	g	h	i	j	k
number of strings	1	6	2	5	7	3	4	10	20	30	100
black beads	4	24	8	20	28	12	16	40	80	120	400
white beads	5	30	10	25	35	15	20	50	100	150	500
total no. of beads	9	54	18	45	63	27	36	90	180	270	900

Another arrangement was 6 white beads between 3 black beads on one end and 5 black ones on the other.

2. Provide the missing information in the record below:

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
number of strings	1	2	3	5	8	10	18	28	20	100
black beads	8	16	24	40	64	80	144	224	160	800
white beads	6	12	18	30	48	60	108	168	120	600
total no. of beads	14	28	42	70	112	140	252	392	280	1,400

Rearranging Letters

3. How many sense or nonsense words can you make by rearranging these letters: T, E, A?

TEA TAE AET
ATE EAT *ETA

How many words are familar? 3

How many words are not familiar words? 3

4. How many sense and nonsense words can you make by rearranging these letters: M, E, A, T?

MTEA TMEA TEMA (TEAM)

(MATE) AMTE ATME ATEM
(MEAT) EMAT EAMT EATM
MTAE TMAE (TAME) TAEM
MAET AMET AEMT AETM
@META EMTA ETMA ETAM

How many words are familar? 4

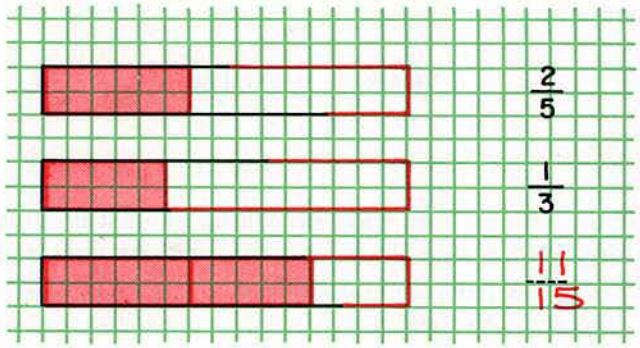
How many words are not familiar words? 20

*If pupils are familiar with the Greek alphabet they may answer 4 familiar words and 2 not familiar words. @ If pupils are familiar with the prefix meta they may answer 5 familiar and 19 not familiar.

"I like the sketch idea and I also like Jean's way of reporting the results. Let's try some more," Al suggested. Everyone agreed.

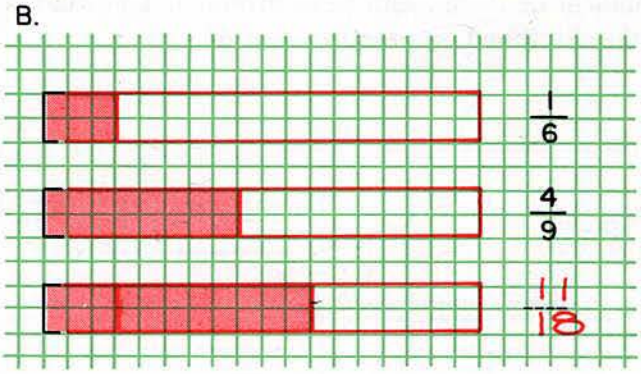
Mrs. Brown provided everyone with graph paper and suggested these problems. Please finish the sketches and the arithmetic they suggest:

A.



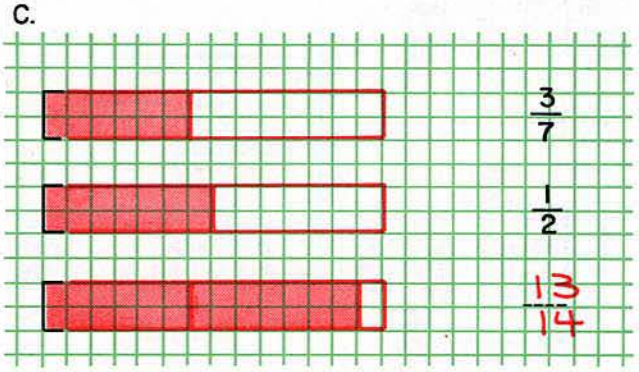
$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$

B.



$\frac{1}{6} + \frac{4}{9} = \frac{3}{18} + \frac{8}{18} = \frac{11}{18}$

C.



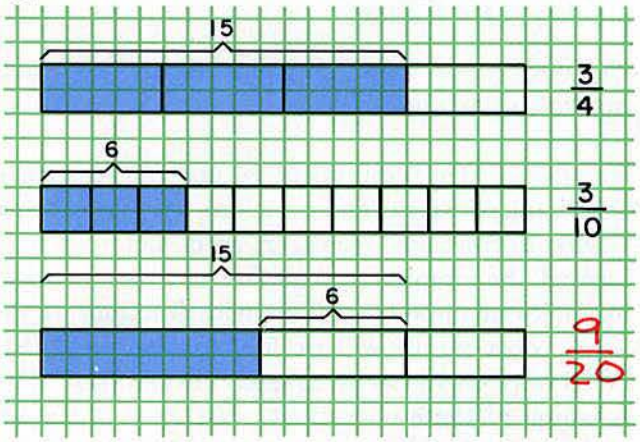
$\frac{3}{7} + \frac{1}{2} = \frac{6}{14} + \frac{7}{14} = \frac{13}{14}$

"I like this method even better than before," Al exclaimed. "I can do any addition and subtraction problems without thinking of a special set of Ben's cards."

"How do you do subtraction?" Alice wanted to know . . . "like $\frac{3}{4} - \frac{3}{10} =$ ____."

"Just draw it out," Al said — and proceeded to show a sketch.

D.



$\frac{3}{4} - \frac{3}{10} = \frac{15}{20} - \frac{6}{20} = \frac{9}{20}$

Mrs. Brown urged her class to use whatever method they felt surest with. She gave them some examples to try:

1. $\frac{1}{6} + \frac{3}{4} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$
2. $\frac{1}{7} + \frac{2}{3} = \frac{3}{21} + \frac{14}{21} = \frac{17}{21}$
3. $\frac{5}{9} + \frac{1}{6} = \frac{10}{18} + \frac{3}{18} = \frac{13}{18}$
4. $\frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$
5. $\frac{7}{8} - \frac{1}{2} = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$
6. $\frac{1}{4} + \frac{2}{7} = \frac{7}{28} + \frac{8}{28} = \frac{15}{28}$

Tasks: In exercises A-D pupils use Al's and Jean's method for performing the addition of fractions and recording results. In exercises 1-6 pupils compute the indicated sums and differences.

Purpose: To continue exploring a rationale for the addition and subtraction of fractions.

Unifying Ideas: Structure; addition and subtraction.

The Lesson: It is natural for teachers accustomed to teaching traditional arithmetic at this point to feel an urge to provide their pupils with rules for adding or subtracting fractions. This urge is fostered by traditional programs, which insist that in the early stages of learning pupils need to know the rules for the best procedures used by those highly competent in arithmetic computation. On the contrary, such premature focusing on the rules leads pupils to think that the rules are the goal we were after all the time. Later, when confronted with a problem, they try to remember the rules—and try to remember the rule for choosing the right rule. All too often, their memory fails them—they misuse the rules or they give up the game as hopeless, realizing they have forgotten the rules.

If, however, stress is laid on the pupil's ability to think through a problem facing him, he will work through to a sensible solution, using rules only when he is confident of them, knowing full well they are only shortcuts that he can test if he has any question.

The general sketching process explored on pupil pages 84 and 85 is as close as we will come to any generalizations or rules at this level in the MATH WORKSHOP program. Such work as pupils have been doing up to this point is quite adequate to help them solve quickly the examples involving addition and subtraction of fractions included in standardized tests through the sixth-grade level. Interestingly enough, such problems are not solved correctly by a significant number of sixth graders.

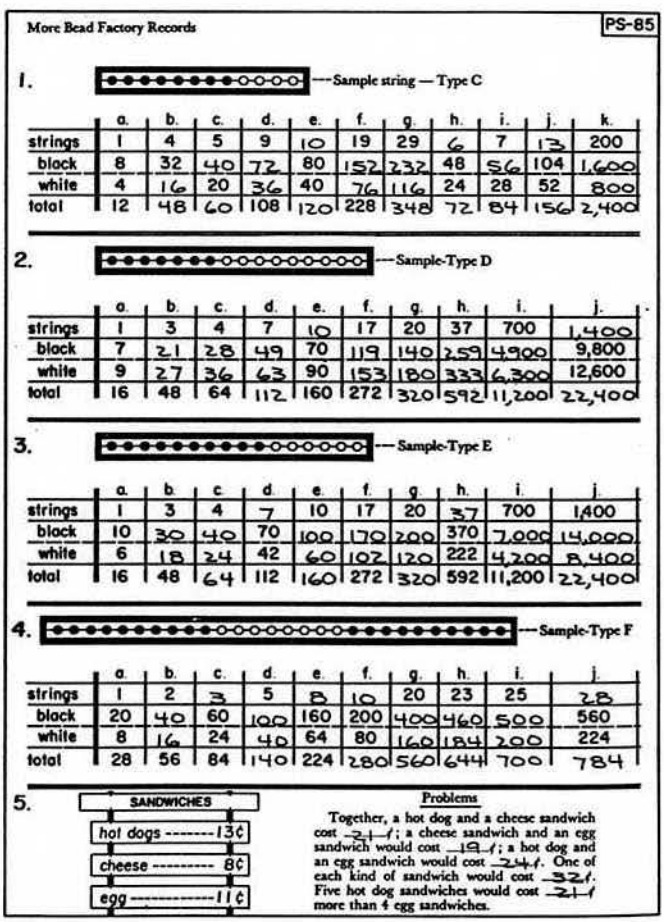
Of course, if you disagree with our decision to stop short of rules at this time, continue as far as you feel you ought to go.

Remember that at this point we do not expect pupils to have achieved mastery in adding and subtracting fractions, though some of them will have developed a high degree of skill. We will, of course, come back to the subject often in Levels E and F in many different contexts.

Bead Factory Arithmetic on practice sheet 85 is but another approach to the system of regrouping we use to make little problems out of big ones in multiplication. Nineteen strings of 12 beads—8 black and 4 white—contain 228 beads. Two reliable ways to arrive at this result are:

19	19	152		19	19	190
×8	×4	+76	and	×10	×2	+38
152	76	228		190	38	228

Those are only two of the ways regrouping can help. We usually choose the second one as a general method, since it leads to simple computations.



More Bead Factory Records — PS-85

1. Sample string — Type C

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.
strings	1	4	5	9	10	19	29	6	7	13	200
black	8	32	40	72	80	152	232	48	56	104	1,600
white	4	16	20	36	40	76	116	24	28	52	800
total	12	48	60	108	120	228	348	72	84	156	2,400

2. Sample-Type D

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
strings	1	3	4	7	10	17	20	37	700	1,400
black	7	21	28	49	70	119	140	259	4,900	9,800
white	9	27	36	63	90	153	180	333	6,300	12,600
total	16	48	64	112	160	272	320	592	11,200	22,400

3. Sample-Type E

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
strings	1	3	4	7	10	17	20	37	700	1,400
black	10	30	40	70	100	170	200	370	7,000	14,000
white	6	18	24	42	60	102	120	222	4,200	8,400
total	16	48	64	112	160	272	320	592	11,200	22,400

4. Sample-Type F

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.
strings	1	2	3	5	8	10	20	23	25	28
black	20	40	60	100	160	200	400	460	500	560
white	8	16	24	40	64	80	160	184	200	224
total	28	56	84	140	224	280	560	644	700	784

5. SANDWICHES

hot dogs ------- 13¢

cheese --------- 8¢

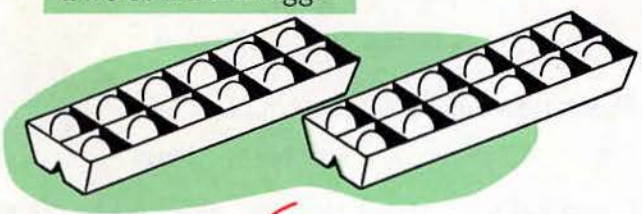
egg ------------ 11¢

Problems

Together, a hot dog and a cheese sandwich cost 21¢; a cheese sandwich and an egg sandwich would cost 19¢; a hot dog and an egg sandwich would cost 24¢. One of each kind of sandwich would cost 32¢. Five hot dog sandwiches would cost 21¢ more than 4 egg sandwiches.

Familiar situations — and the arithmetic they suggest.

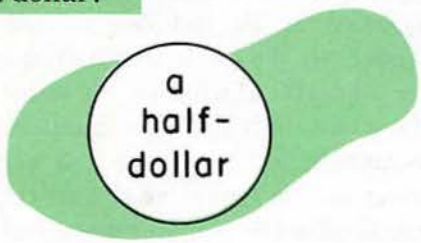
A. Two of a dozen eggs:



Two of 12 eggs is 6 eggs.

$2 \times 12 = 24$ and $12 \times 2 = 24$

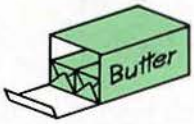
B. Half a dollar:



Half of 100 cents is 50 cents.

$\frac{1}{2} \times 100 = 50$ and $100 \times \frac{1}{2} = 50$

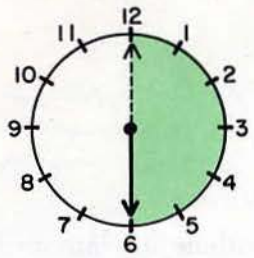
C. Half a pound:



Half of 16 ounces is 8 ounces.

$\frac{1}{2} \times 16 = 8$ and $16 \times \frac{1}{2} = 8$

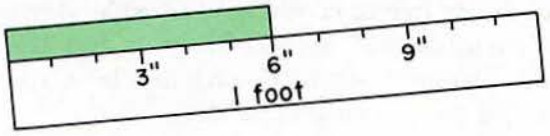
D. Half an hour:



Half of 60 minutes is 30 min.

$\frac{1}{2} \times 60 = 30$ and $60 \times \frac{1}{2} = 30$

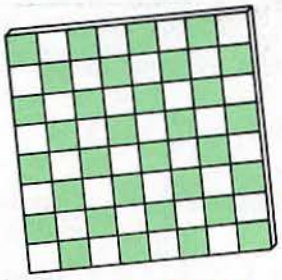
E. Half a foot:



Half of 12 inches is 6 inches.

$\frac{1}{2} \times 12 = 6$ and $12 \times \frac{1}{2} = 6$

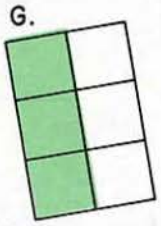
F. Half of the spaces on a checkerboard:



Half of 64 spaces is 32 spaces.

$\frac{1}{2} \times 64 = 32$ and $64 \times \frac{1}{2} = 32$

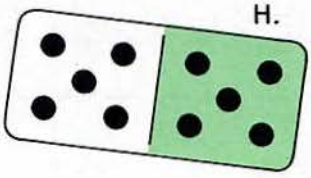
G.



$\frac{1}{2} \times 6 = 3$ and $6 \times \frac{1}{2} = 3$

$\frac{1}{2} \times 10 = 5$ and $10 \times \frac{1}{2} = 5$

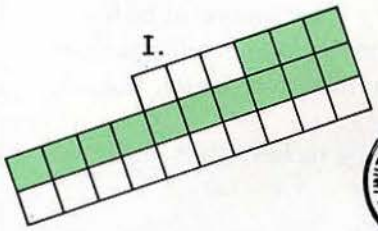
H.



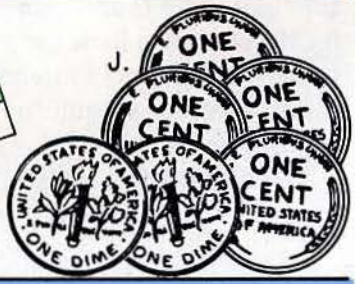
$\frac{1}{2} \times 26 = 13$ and $26 \times \frac{1}{2} = 13$

$\frac{1}{2} \times 24 = 12$ and $24 \times \frac{1}{2} = 12$

I.



J.



86-87 Exploring situations in which this idea has a familiar setting

Tasks: In exercises A-F pupils use the clue provided by each picture to make each pair of sentences true. In exercises G-J pupils match the situation to the pair of incomplete sentences it suggests, and make each pair of sentences true.

Purpose: To introduce the multiplication of a whole number and a fraction.

Unifying Ideas: Structure; number and counting; multiplication and division.

The Lesson: On pupil pages 86 and 87 we suggest that multiplication (and division) may later be extended to fractions. However, we do not wish to go beyond the simplest translation of a few common examples of these operations with fractions at this level. Most pupils will profit from more sophisticated work only after an extensive exploration of the meaning of these operations. Just as you can add and subtract things, you can add and subtract fractions of things. But, just as you cannot multiply things by things, you cannot multiply fractions of things by fractions of things. The following expressions have no meaning:

$$\frac{1}{2} \text{ dollar} \times \frac{1}{2} \text{ dollar}$$

$$\frac{1}{3} \text{ pie} \times \frac{1}{4} \text{ pie}$$

However, just as we multiply things by numbers, we can multiply fractions of things by numbers:

$$\frac{1}{2} \text{ dollar} \times 3 = 1\frac{1}{2} \text{ dollars}$$

$$\frac{1}{3} \text{ pie} \times 5 = 1\frac{2}{3} \text{ pies}$$

And, although we cannot multiply things by fractions of things, we can multiply things by fractions, since fractions are numbers:

$$1 \text{ dozen eggs} \times 2 = 2 \text{ dozen eggs}$$

$$1 \text{ dozen eggs} \times \frac{1}{2} = \frac{1}{2} \text{ dozen eggs} = 6 \text{ eggs}$$

$$1 \text{ pound} \times 2 = 2 \text{ pounds}$$

$$1 \text{ pound} \times \frac{1}{2} = \frac{1}{2} \text{ pound} = 8 \text{ ounces}$$

Although we prefer writing the number after the multiplication sign, as above, the way we describe things in everyday language leads us to write

$$\frac{1}{2} \times 1 \text{ pound} = 8 \text{ ounces}$$

to show that half of $(\frac{1}{2} \times)$ 1 pound is $(=)$ 8 ounces.

On pupil pages 86 and 87, we write expressions both ways.

Pupils can make up many additional examples to illustrate the idea of "half of." They might also notice that:

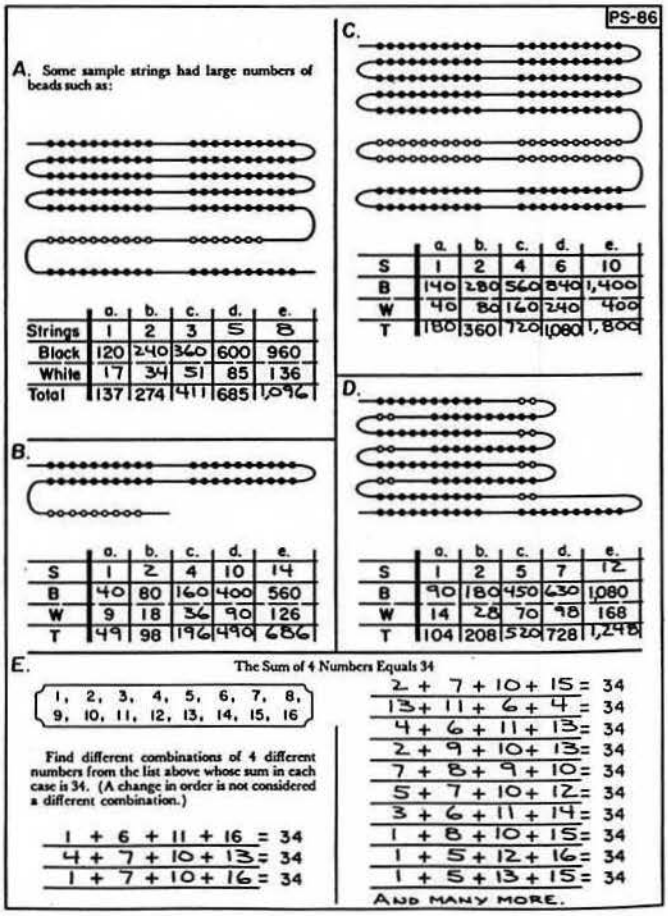
$$1 \text{ foot} \times \frac{1}{2} = 6 \text{ inches}$$

and

$$1 \text{ foot} \div 2 = 6 \text{ inches}$$

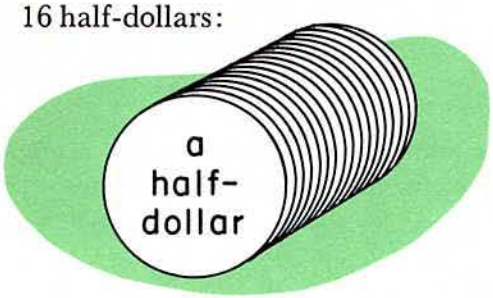
Thus, we have two ways of talking about the same thing, and a seed is planted that will be nurtured later into a general method of dividing by a fraction.

On practice sheet 86 we consider longer strings of beads and indicate a clear preference for regrouping as often as possible in multiples of 10. The strings are getting so long that some shorthand method of indicating them ought to be invented. You will find the shorthand on practice sheet 87.



A.

16 half-dollars:

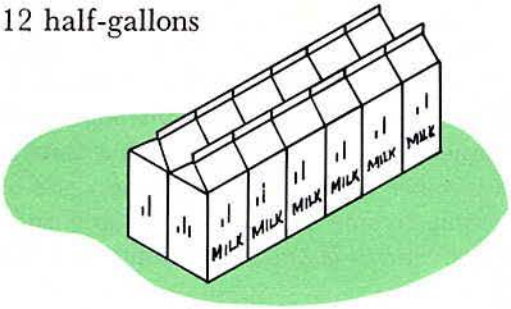


16 half-dollars is 8 dollars

$16 \times \frac{1}{2} = 8$ and $\frac{1}{2} \times 16 = 8$

B.

12 half-gallons

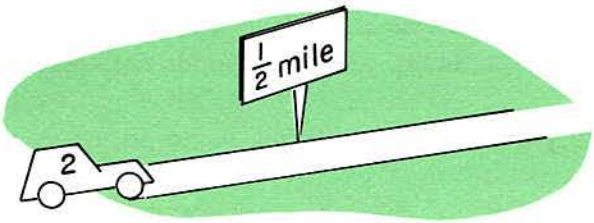


12 half-gallons is 6 gallons

$12 \times \frac{1}{2} = 6$ and $\frac{1}{2} \times 12 = 6$

C.

100 half-miles:

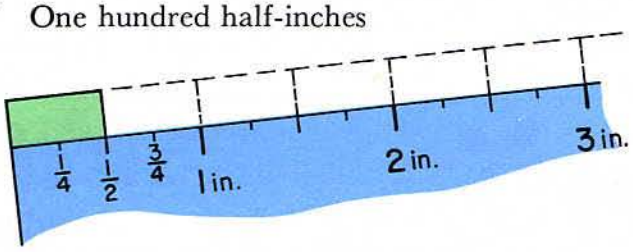


100 half-miles is 50 miles

$100 \times \frac{1}{2} = 50$ and $\frac{1}{2} \times 100 = 50$

D.

One hundred half-inches

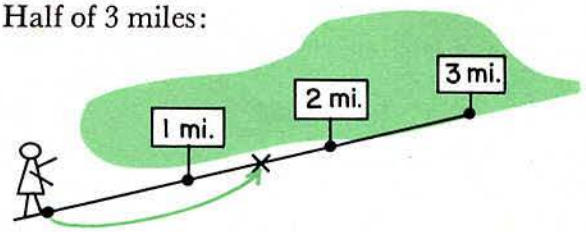


100 half-inches is 50 inches

$100 \times \frac{1}{2} = 50$ and $\frac{1}{2} \times 100 = 50$

E.

Half of 3 miles:

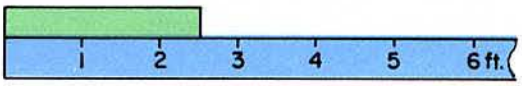


Half of 3 miles is a mile and a half.

$\frac{1}{2} \times 3 = 1\frac{1}{2}$ and $3 \times \frac{1}{2} = 1\frac{1}{2}$

F.

Half of 5 feet:

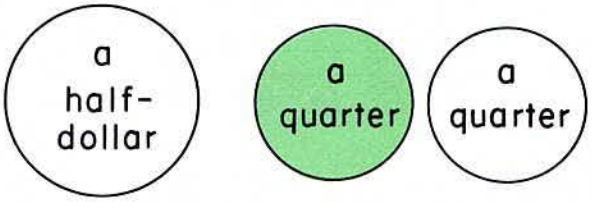


Half of 5 feet is two and a half feet.

$\frac{1}{2} \times 5 = 2\frac{1}{2}$ and $5 \times \frac{1}{2} = 2\frac{1}{2}$

G.

Half of a half-dollar:

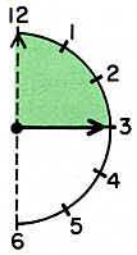


Half of a half-dollar is a quarter.

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

H.

Half of a lunch hour:



Half of a half-hour is a quarter of an hour.

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Tasks: *In exercises A-F pupils use the clues provided in each picture to make each pair of sentences true. In exercise H pupils make true the sentence suggested by the situation.*

Purpose: To continue an introduction of multiplication of fractions and whole numbers. To introduce the idea of multiplying fractions.

Unifying Ideas: Structure; multiplication and division.

The Lesson: While the examples on pupil page 87 are limited almost entirely to the idea of "half of," feel perfectly free to continue your explorations. In what circumstances are we likely to use the expression "a third of" or "multiplied by a third"? We would have little trouble with:

$$1 \text{ foot} \times \frac{1}{3} = \underline{\quad} \text{ inches}$$

$$75¢ \times \frac{1}{3} = \underline{\quad}¢$$

$$1 \text{ dozen eggs} \times \frac{1}{3} = \underline{\quad} \text{ eggs}$$

$$1 \text{ day} \times \frac{1}{3} = \underline{\quad} \text{ hours}$$

There is more of a question in the following:

$$1 \text{ dollar} \times \frac{1}{3} = \underline{?}¢$$

$$1 \text{ pound} \times \frac{1}{3} = \underline{?} \text{ ounces}$$

$$1 \text{ week} \times \frac{1}{3} = \underline{?} \text{ days}$$

The class might be asked for possible ways out of these predicaments. They might suggest: A little more than 33¢, or even 33¢ and ⅓ of a penny for the first; more than 4 ounces but less than 5 ounces for the second; a little more than 2 days, or 56 hours for the last. Be more interested in many suggestions rather than in specific numbers of cents, ounces, days, or hours.

Further, if 1 foot × ⅓ = 4 inches, what might we mean by:

$$1 \text{ foot} \times \frac{2}{3} = \underline{\quad} \text{ in.}$$

About the only suggestion that follows the general lines of our discussion would lead to 8 inches as the answer.

$$1 \text{ foot} \times \frac{1}{3} = 4 \text{ inches}$$

$$4 \text{ inches} \times 2 = 8 \text{ inches}$$

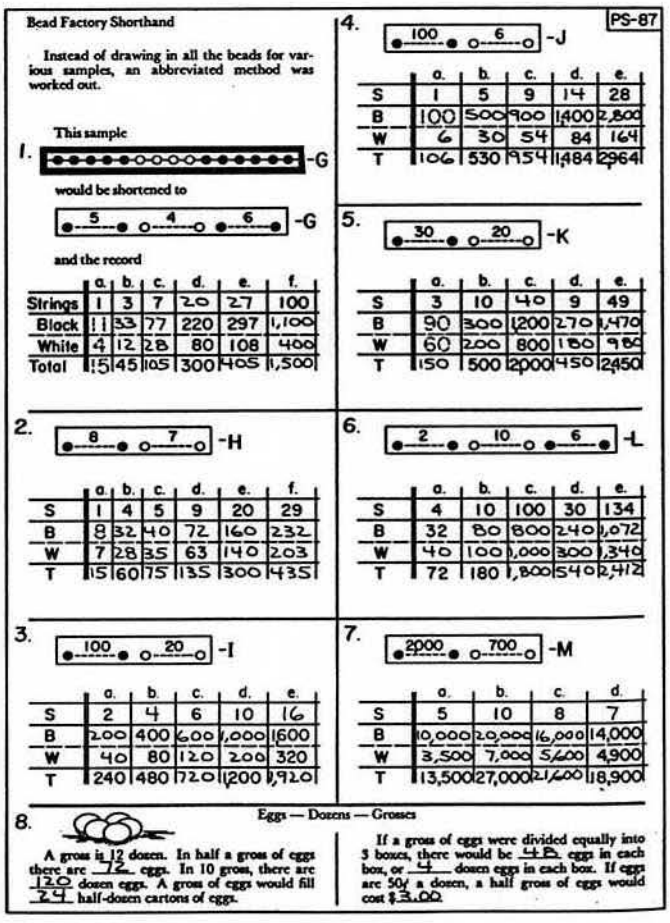
Both steps follow our pattern of:

$$\text{things} \times \text{a number} = \text{things}$$

In exploring ⅔ of a dollar, etc., push on as far as you like. We would, at this point, strongly urge you not to push for generalizations or to introduce any rules. These two pupil pages are more than adequate explorations of ideas that need to be studied in depth—and they will be in Levels E and F.

The scantest of hints about developments ahead is suggested in the last two related examples on pupil page 87. We know that half of a half dollar is 25¢, or a quarter of a dollar. We know that half of a half hour is 15 minutes, or a quarter of an hour. It seems quite logical to assume that a half of a half of anything will be a quarter of that thing.

Practice sheet 87 contains more drill in the setting of Bead Factory Arithmetic.


Bead Factory Shorthand
PS-87
Instead of drawing in all the beads for various samples, an abbreviated method was worked out.
This sample
-G
would be shortened to
5 4 6 -G
and the record
1. -G
2. 8 7 -H
3. 100 20 -I
4. 100 6 -J
5. 30 20 -K
6. 2 10 6 -L
7. 2000 700 -M
8. Eggs — Dozens — Grosses
A gross is 12 dozen. In half a gross of eggs there are 72 eggs. In 10 gross, there are 120 dozen eggs. A gross of eggs would fill 24 half-dozen cartons of eggs.
If a gross of eggs were divided equally into 3 boxes, there would be 48 eggs in each box, or 4 dozen eggs in each box. If eggs are 50¢ a dozen, a half gross of eggs would cost $3.00

A Game of FEWEST NUMBERS

In the green blocks below, please write one of the numbers written in the large green block. All sentences are to be true statements.

0 , 1 , 2 , 4 , 8 , 16

a. 5 x 4 = 20

b. 10 x 8 = 80

c. 3 x 16 = 48

d. 7 x 4 = 28

e. 8 x 8 = 64

f. 9 x 4 = 36

g. 14 x 1 = 14

h. 17 x 0 = 0

You cannot complete the following if you follow the rules

i. 7 x ____ = 21

j. 4 x ____ = 28

. . . because 3 and 7 are not listed in the large green block above. In such cases, we use a 2- or 3-step solution:

k.
7 x 2 = 14
7 x 1 = 7
7 x 3 = 21

l.
4 x 4 = 16
4 x 2 = 8
4 x 1 = 4
4 x 7 = 28

In such cases, the purpose of the game is to use the fewest numbers from the large green block — to find the solution with the fewest steps.

Please complete the blanks in the next examples so that all completed sentences are true statements:

A.
8 x 4 = 32
8 x 1 = 8
8 x 4 = 32
8 x 9 = 72

B.
12 x 4 = 48
12 x 2 = 24
12 x 2 = 24
12 x 8 = 96

These are "losing solutions" because A has a 2-step solution and B has a 1-step solution. Do you know what they are?

A.
8 x 8 = 64
8 x 1 = 8
8 x 9 = 72

B.
12 x 8 = 96

Please complete the following so that each sentence is a true statement. You will need to use all the steps indicated by the green blocks.

C.
5 x 2 = 10
5 x 8 = 40
5 x 10 = 50

D.
9 x 4 = 36
9 x 2 = 18
9 x 6 = 54

E.
3 x 16 = 48
3 x 4 = 12
3 x 1 = 3
3 x 21 = 63

F.
6 x 8 = 48
6 x 4 = 24
6 x 2 = 12
6 x 1 = 6
6 x 15 = 90

Tasks: In all exercises on this pupil page only numbers from the given list may be used to fill green blocks. Other numbers may be used to fill blanks. Exercises i *and* j *contain sentences that cannot be completed according to these rules. The remaining exercises show a strategy for using the given numbers to find numbers for these sentences. Preference is shown for "fewest number" solutions.*

Purpose: To use the distributive principle of multiplication over addition for solving equations.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: The work on pupil page 88 is closely related to the *Punch Card Arithmetic* developed earlier in the practice sheets. The problems might be given in the following form, with directions to circle a combination of numbers whose sum would make the sentence into a true statement:

5 × [1 , 2 , ④ , 8 , 16] = 20

7 × [①, ②, 4 , 8 , 16] = 21

6 × [①, ②, ④, ⑧, 16] = 90

There is a bit of a puzzle—finding the right combinations to punch. And the numbers are selected so that there is always a unique solution if the "fewest numbers" rule is followed.

This activity provides quite a lot of practice in multiplication and addition. All examples involve the idea of regrouping—part of the process of making small problems out of larger problems.

Practice sheet 88 is also designed for extensive practice in multiplication and addition—finding the rules from a limited number of items in a sequence and then extending that sequence.

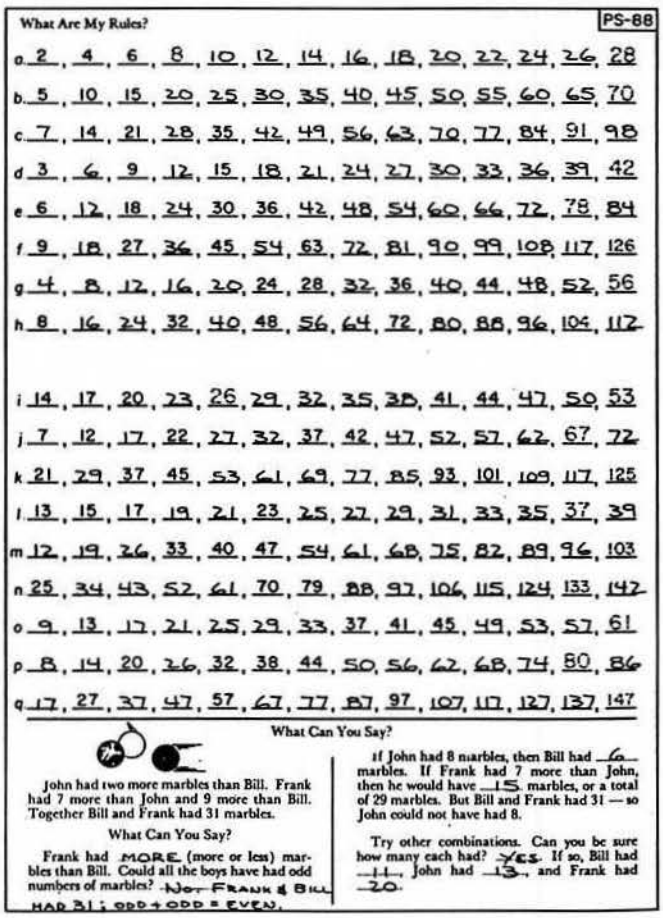
For those pupils who need the challenge of mental arithmetic more than practice, ask them to leave all entries blank except the tenth and eleventh (counting from the left) or any other particular terms. Front runners may find shortcuts that will enable them to skip from any term to another term a known number of steps away. For example, in sequence *a*, the tenth and eleventh terms must be:

$$2\times10=20$$
$$\text{and } 2\times11=22.$$

The same kind of shortcut will help in sequences *b* through *h*. The story changes with sequence *i*. One way to find the tenth and eleventh terms directly would be to see that they must be:

$$14+[3\times(10-1)]=41$$
$$\text{and } 14+[3\times(11-1)]=44.$$

As soon as you know one term and the difference between adjacent terms (it is constant throughout each sequence), you can quickly find any term a known number of steps away to the right or left. Urge pupils to find a shortcut . . . and to use it. Finding shortcuts is the most important business of arithmetic.



What Are My Rules? PS-88

a. 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28

b. 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70

c. 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98

d. 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42

e. 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84

f. 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126

g. 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56

h. 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, 112

i. 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53

j. 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67, 72

k. 21, 29, 37, 45, 53, 61, 69, 77, 85, 93, 101, 109, 117, 125

l. 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39

m. 12, 19, 26, 33, 40, 47, 54, 61, 68, 75, 82, 89, 96, 103

n. 25, 34, 43, 52, 61, 70, 79, 88, 97, 106, 115, 124, 133, 142

o. 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61

p. 8, 14, 20, 26, 32, 38, 44, 50, 56, 62, 68, 74, 80, 86

q. 17, 27, 37, 47, 57, 67, 77, 87, 97, 107, 117, 127, 137, 147

What Can You Say?

John had two more marbles than Bill. Frank had 7 more than John and 9 more than Bill. Together Bill and Frank had 31 marbles.

What Can You Say?

Frank had MORE (more or less) marbles than Bill. Could all the boys have had odd numbers of marbles? NO. FRANK & BILL HAD 31; ODD + ODD = EVEN.

If John had 8 marbles, then Bill had 6 marbles. If Frank had 7 more than John, then he would have 15 marbles, or a total of 29 marbles. But Bill and Frank had 31 — so John could not have had 8.

Try other combinations. Can you be sure how many each had? YES. If so, Bill had 11, John had 13, and Frank had 20.

Another Version of FEWEST NUMBERS

The changes in the rules are:

(1) Numbers for the blue blocks must be selected from the large blue block.

(2) No number can be selected more than once in any example.

0 , 1 , 2 , 3 , 6 , 9 , 18

When you do not need one of the steps use "0" in the blue block, and that step will not count as a step.

1 | 2

8 x 9 = 72
8 x 3 = 24
8 x 1 = 8
8 x 13 = 104

9 x 6 = 54
9 x 1 = 9
9 x 0 = 0
9 x 7 = 63

A 2-step solution

3

7 x 9 = 63
7 x 2 = 14
7 x 0 = 0
7 x 11 = 77

This is a 2-step solution

4

8 x 9 = 72
8 x 3 = 24
8 x 12 = 96

5

5 x 9 = 45
5 x 6 = 30
5 x 15 = 75

6

6 x 9 = 54
6 x 3 = 18
6 x 2 = 12
6 x 14 = 84

7

12 x 6 = 72
12 x 0 = 0
12 x 6 = 72

This is a one-step solution

8 | 9

2 x 18 = 36
2 x 9 = 18
2 x 3 = 6
2 x 1 = 2
2 x 31 = 62

13 x 9 = 117
13 x 6 = 78
13 x 2 = 26
13 x 0 = 0
13 x 17 = 221

10 | 11

15 x 9 = 135
15 x 6 = 90
15 x 1 = 15
15 x 16 = 240

3 x 18 = 54
3 x 6 = 18
3 x 24 = 72

12

4 x 18 = 72
4 x 9 = 36
4 x 6 = 24
4 x 33 = 132

13

21 x 9 = 189
21 x 3 = 63
21 x 12 = 252

14 | 15

5 x 18 = 90
5 x 1 = 5
5 x 0 = 0
5 x 19 = 95

15 x 9 = 135
15 x 3 = 45
15 x 1 = 15
15 x 13 = 195

16 | 17

10 x 18 = 180
10 x 9 = 90
10 x 3 = 30
10 x 2 = 20
10 x 32 = 320

9 x 18 = 162
9 x 6 = 54
9 x 1 = 9
9 x 0 = 0
9 x 25 = 225

89 Exploring the games using another special set of numbers - 0, 1, 2, 3, 6, 9, 18

Tasks: In each exercise pupils use the new rules to make true sentences.

Purpose: To use the distributive principle of multiplication over addition for solving equations.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: As the list of numbers is changed to 0, 1, 2, 3, 6, 9, 18 on pupil page 89, we are again reminded of *Punch Card Arithmetic*, where this same list was often used. If you were asked to list all whole numbers expressed as the sum of the fewest numbers from this list, you would write:

0 = 0	6 = 6	12 = 9 + 3
1 = 1	7 = 6 + 1	13 = 9 + 3 + 1
2 = 2	8 = 6 + 2	14 = 9 + 3 + 2
3 = 3	9 = 9	15 = 9 + 6
4 = 3 + 1	10 = 9 + 1	16 = 9 + 6 + 1
5 = 3 + 2	11 = 9 + 2	17 = 9 + 6 + 2

And so on, up to 39 = 18 + 9 + 6 + 3 + 2 + 1. Suppose that you were asked to do the same thing by circling the fewest numbers below.

9	3	1
9	3	1

For each 2 that you used in the first list, you would circle two 1's. For each 6, you would circle two 3's, and, for each 18, two 9's. Let's indicate how many of each of these numbers must be used to obtain the required sum:

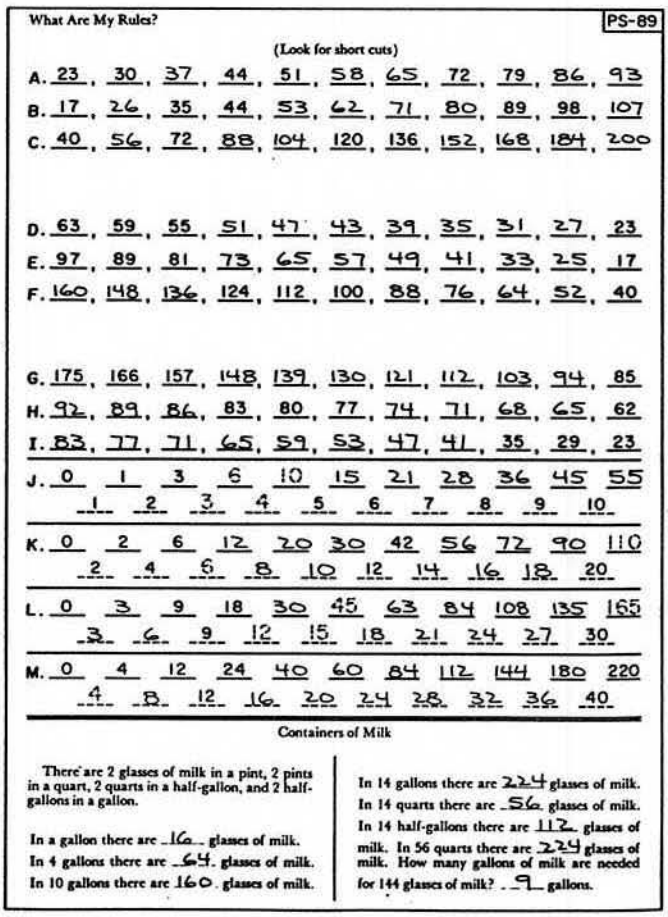
(9)(3)(1)	(9)(3)(1)	(9)(3)(1)
0 = 0, 0, 0	6 = 0, 2, 0	12 = 1, 1, 0
1 = 0, 0, 1	7 = 0, 2, 1	13 = 1, 1, 1
2 = 0, 0, 2	8 = 0, 2, 2	14 = 1, 1, 2
3 = 0, 1, 0	9 = 1, 0, 0	15 = 1, 2, 0
4 = 0, 1, 1	10 = 1, 0, 1	16 = 1, 2, 1
5 = 0, 1, 2	11 = 1, 0, 2	17 = 1, 2, 2

Now let's make a list of those whole numbers, starting with 0, whose decimal numeral is some combination of the digits 0, 1, or 2:

Start: 0	6th: 20	12th: 110
1st: 1	7th: 21	13th: 111
2nd: 2	8th: 22	14th: 112
3rd: 10	9th: 100	15th: 120
4th: 11	10th: 101	16th: 121
5th: 12	11th: 102	17th: 122

Compare this list with both of those above. We are utilizing the pattern of a system of numeration as it would have developed if we limited ourselves to the digits 0 through 2 rather than the digits 0 through 9. This is a base-three system of numeration, rather than the base-ten, or decimal, system of numeration we usually use.

Exercises A through I of practice sheet 89 are a further exploration of constant-difference sequences. In J through M we begin exploring sequences in which the differences between adjacent terms are not constant, but have a definite pattern of their own.



What Are My Rules? PS-89

(Look for short cuts)

A. 23, 30, 37, 44, 51, 58, 65, 72, 79, 86, 93

B. 17, 26, 35, 44, 53, 62, 71, 80, 89, 98, 107

C. 40, 56, 72, 88, 104, 120, 136, 152, 168, 184, 200

D. 63, 59, 55, 51, 47, 43, 39, 35, 31, 27, 23

E. 97, 89, 81, 73, 65, 57, 49, 41, 33, 25, 17

F. 160, 148, 136, 124, 112, 100, 88, 76, 64, 52, 40

G. 175, 166, 157, 148, 139, 130, 121, 112, 103, 94, 85

H. 92, 89, 86, 83, 80, 77, 74, 71, 68, 65, 62

I. 83, 77, 71, 65, 59, 53, 47, 41, 35, 29, 23

J. 0 1 3 6 10 15 21 28 36 45 55
1 2 3 4 5 6 7 8 9 10

K. 0 2 6 12 20 30 42 56 72 90 110
2 4 6 8 10 12 14 16 18 20

L. 0 3 9 18 30 45 63 84 108 135 165
3 6 9 12 15 18 21 24 27 30

M. 0 4 12 24 40 60 84 112 144 180 220
4 8 12 16 20 24 28 32 36 40

Containers of Milk

There are 2 glasses of milk in a pint, 2 pints in a quart, 2 quarts in a half-gallon, and 2 half-gallons in a gallon.

In a gallon there are 16 glasses of milk.
In 4 gallons there are 64 glasses of milk.
In 10 gallons there are 160 glasses of milk.

In 14 gallons there are 224 glasses of milk.
In 14 quarts there are 56 glasses of milk.
In 14 half-gallons there are 112 glasses of milk. In 56 quarts there are 224 glasses of milk. How many gallons of milk are needed for 144 glasses of milk? 9 gallons.

A Third Version of **FEWEST NUMBERS**

(1) Numbers for the green blocks must be selected from the large green block.

(2) No number (except 0) in this list can be used more than once in any example.

0 , 1 , 2 , 3 , 4 , 5 , 10 , 15
20 , 25 , 50 , 75 , 100 , 125

1

14 x 10 = 140
14 x 3 = 42
14 x 13 = 182

2

8 x 15 = 120
8 x 2 = 16
8 x 17 = 136

3

5 x 100 = 500
5 x 15 = 75
5 x 3 = 15
5 x 118 = 590

4

18 x 20 = 360
18 x 3 = 54
18 x 0 = 0
18 x 23 = 414

5

10 x 25 = 250
10 x 20 = 200
10 x 2 = 20
10 x 47 = 470

6

23 x 10 = 230
23 x 4 = 92
23 x 0 = 0
23 x 14 = 322

7

8 x 75 = 600
8 x 10 = 80
8 x 4 = 32
8 x 89 = 712

8

31 x 5 = 155
31 x 2 = 62
31 x 0 = 0
31 x 7 = 217

9

24 x 10 = 240
24 x 3 = 72
24 x 13 = 312

10

15 x 5 = 75
15 x 4 = 60
15 x 9 = 135

11

19 x 20 = 380
19 x 1 = 19
19 x 21 = 399

12

8 x 50 = 400
8 x 15 = 120
8 x 65 = 520

13

15 x 10 = 150
15 x 3 = 45
15 x 13 = 195

14

13 x 15 = 195
13 x 4 = 52
13 x 19 = 247

15

25 x 25 = 625
25 x 2 = 50
25 x 27 = 675

16

27 x 25 = 675
27 x 4 = 108
27 x 29 = 783

17

30 x 15 = 450
30 x 0 = 0
30 x 15 = 450

18

31 x 5 = 155
31 x 3 = 93
31 x 8 = 248

19

12 x 25 = 300
12 x 20 = 240
12 x 4 = 48
12 x 49 = 588

20

42 x 20 = 840
42 x 3 = 126
42 x 0 = 0
42 x 23 = 966

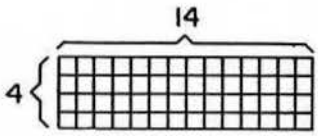
90 Exploring the games using another special set of numbers - 0, 1, 2, 3, 4, 5, 10, 15, etc.

Tasks: In each exercise pupils use the rules and the new list of numbers to make true sentences.

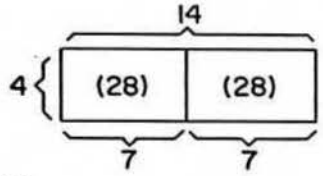
Purpose: To use the distributive principle of multiplication over addition to solve equations.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: The game of Fewest Numbers continues on pupil page 90 with a new list of numbers. This activity is still another attempt to help pupils see the many different ways a hard problem can be broken up into more manageable parts—and the parts put back together again. How many tiles?

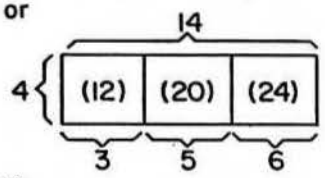


Since that is probably not one of the facts we have in our memory, we break it down into one of these:



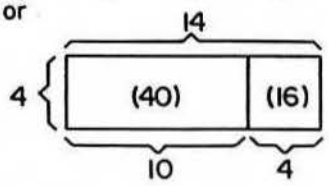
28
+28
56

or



12
20
+24
56

or

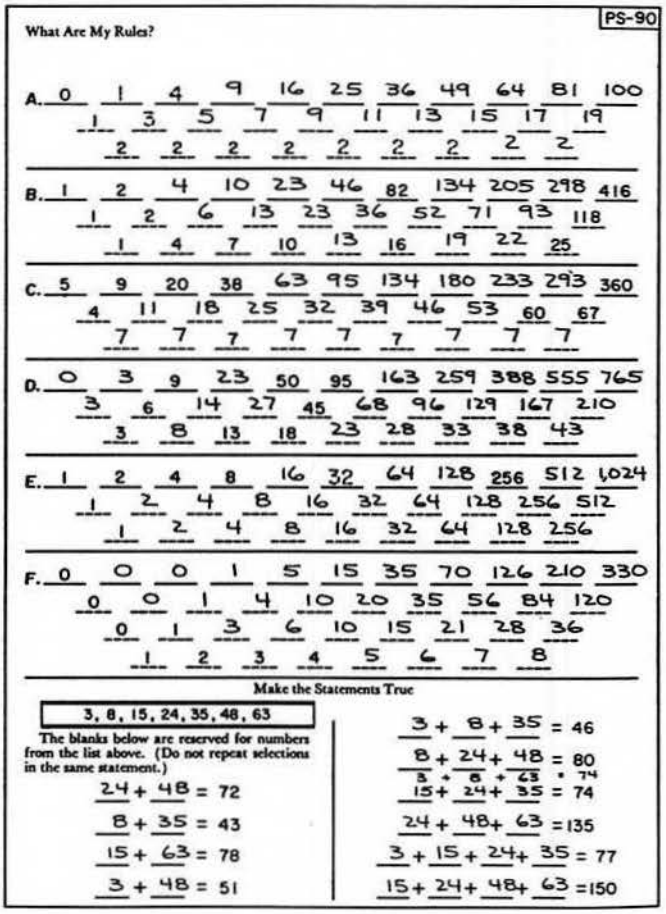


40
+16
56

In each case, the "big" problem is broken into two or three parts—all of which are simple multiplication facts. Then we add the products of the parts to get the total number of tiles.

Because we will eventually be concerned with a general system for breaking into parts, we recognize the last example as having certain procedural advantages. But it is of fundamental importance that pupils realize that the other methods are just as "right," because they lead reliably to correct answers. The pupil who knows several ways to get the correct answer understands what he is doing. The pupil who knows only a single way is a slave to a rule.

What Are My Rules? on practice sheet 90 will give pupils some insight into how rather complicated sequences are generated. While it is included here to provide computational practice for those who need it, this is only the beginning of an acquaintance with a powerful tool for analyzing a collection of data whose inner relationships might otherwise escape the collector or investigator.



PS-90

What Are My Rules?

A. 0 1 4 9 16 25 36 49 64 81 100
1 3 5 7 9 11 13 15 17 19
2 2 2 2 2 2 2 2 2

B. 1 2 4 10 23 46 82 134 205 298 416
1 2 6 13 23 36 52 71 93 118
1 4 7 10 13 16 19 22 25

C. 5 9 20 38 63 95 134 180 233 293 360
4 11 18 25 32 39 46 53 60 67
7 7 7 7 7 7 7 7 7

D. 0 3 9 23 50 95 163 259 388 555 765
3 6 14 27 45 68 96 129 167 210
3 8 13 18 23 28 33 38 43

E. 1 2 4 8 16 32 64 128 256 512 1,024
1 2 4 8 16 32 64 128 256 512
1 2 4 8 16 32 64 128 256

F. 0 0 0 1 5 15 35 70 126 210 330
0 0 1 4 10 20 35 56 84 120
0 1 3 6 10 15 21 28 36
1 2 3 4 5 6 7 8

Make the Statements True

3, 8, 15, 24, 35, 48, 63

The blanks below are reserved for numbers from the list above. (Do not repeat selections in the same statement.)

24 + 48 = 72
8 + 35 = 43
15 + 63 = 78
3 + 48 = 51

3 + 8 + 35 = 46
8 + 24 + 48 = 80
3 + 8 + 63 = 74
15 + 24 + 35 = 74
24 + 48 + 63 = 135
3 + 15 + 24 + 35 = 77
15 + 24 + 48 + 63 = 150

A Fourth Version of FEWEST NUMBERS

(1) Numbers for the blue blocks must be selected from the three large blue blocks.

(2) Only one number (except 0) can be used from each list in each example.

(3) Use 0 as often as you can, to show you used less blocks than are provided.

List A. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

List B. 0, 10, 20, 30, 40, 50, 60, 70, 80, or 90.

List C. 0, 100, 200, 300, 400, . . . 800 or 900.

Remember: You must not use more than one number from a list in each example.

1

12 x 10 = 120
12 x 3 = 36
12 x 13 = 156

2

9 x 10 = 90
9 x 8 = 72
9 x 18 = 162

3

15 x 10 = 150
15 x 7 = 105
15 x 17 = 255

4

13 x 20 = 260
13 x 4 = 52
13 x 24 = 312

5

7 x 100 = 700
7 x 8 = 56
7 x 108 = 756

6

6 x 100 = 600
6 x 40 = 240
6 x 140 = 840

7

26 x 10 = 260
26 x 5 = 130
26 x 15 = 390

8

15 x 20 = 300
15 x 0 = 0
15 x 20 = 300

9

18 x 10 = 180
18 x 8 = 144
18 x 18 = 324

10

13 x 40 = 520
13 x 3 = 39
13 x 43 = 559

11

24 x 30 = 720
24 x 6 = 144
24 x 36 = 864

12

27 x 10 = 270
27 x 9 = 243
27 x 19 = 513

13

17 x 100 = 1700
17 x 10 = 170
17 x 110 = 1870

14

34 x 70 = 2380
34 x 1 = 34
34 x 71 = 2414

15

12 x 100 = 1200
12 x 20 = 240
12 x 7 = 84
12 x 127 = 1524

16

37 x 60 = 2220
37 x 9 = 333
37 x 0 = 0
37 x 69 = 2553

(continued on next page)

Please do the examples below "in your head."

	A.	B.	C.	D.
17	18 x 10 = 180	27 x 4 = 108	27 x 40 = 1080	27 x 400 = 10,800
18	37 x 10 = 370	43 x 8 = 344	43 x 80 = 3440	43 x 800 = 34,400

91-93 Exploring the games using three special sets of numbers - those we use in the quotient of the usual long division algorithm

Tasks: In examples 1-16 pupils use the rules to make true sentences. In examples 17 and 18 pupils perform mental calculations to determine the missing products.

Purpose: To use the distributive principle of multiplication over addition to solve equations.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: The list on pupil page 91 for Fewest Numbers looks more familiar. We have been operating without the advantages place value can provide —advantages that keep the computations as simple as possible. Not only are those advantages now made available, but the numbers are selected for the list in such a way as to make each selection unique.

Exercise 2 is, in effect, one way to find the product of 9×18. Since that fact is not usually memorized, this problem is broken up so each of the parts requires only a memorized fact:

$$\begin{aligned} 9\times10 &= 90 \\ 9\times\ 8 &= 72 \\ \hline 9\times18 &= 162 \end{aligned}$$

There is no other choice for breaking up this problem under the rules of the game of Fewest Numbers as long as we use the specified list. But is this the best way to break up the problem if the rules were relaxed? Probably. But one might reasonably argue that:

$$\begin{aligned} 9\times18 &= 9\times(9\times2) \\ &= (9\times9)\times2 \\ &= 81\times2 \\ &= 162 \end{aligned}$$

has its advantages. For another example, one may claim that the following procedure is clearly superior to the usual one:

$$\begin{aligned} 35\times16 &= (7\times5)\times(2\times8) \\ &= (7\times8)\times(2\times5) \\ &= 56\times10 \\ &= 560 \end{aligned}$$

This procedure is the simplest for mental arithmetic.

Keep a sharp lookout for any lack of confidence your pupils may show when faced with:

$13\times20 =$ ____ $\qquad 17\times100 =$ ____

$26\times10 =$ ____ $\qquad 37\times60 =$ ____

$24\times30 =$ ____

Because of the built-in use of place value in the multiplication algorithm, pupils may go mechanically through the steps required without the faintest notion of how to approach 30×40. This example is completed in the usual algorithm as if it were simply $3\times4=12$ with the result then written in a particular place just because it's open, not because its position gives it a value of 1200. Consider it in this example:

$$\begin{array}{r} 32 \\ \times\ 47 \\ \hline 214 \\ 128 \\ \hline 1494 \end{array}$$

Don't let skill with a mechanical manipulation hide a basic weakness.

On practice sheet 91 the Fibonacci sequence and some related sequences are explored. In a Fibonacci sequence, each term (except the first two) is the sum of the two preceding terms.

PS-91

What is my rule for the sequence?

1 1 2 3 5 8 13 21 34 55 89 144 233 377

1+1=2, 1+2=3, 2+3=5, 3+5=8, 5+8=13, 8+13=21

Here are other sequences with similar rules.

A. 2 1 3 4 7 11 18 29 47 76 123 199

B. 3 1 4 5 9 14 23 37 60 97 157 254

C. 5 2 7 9 16 25 41 66 107 173 280 453

D. 4 3 7 10 17 27 44 71 115 186 301 487

E. 0 2 2 4 6 10 16 26 42 68 110 178

F. 4 1 5 6 11 17 28 45 73 118 191 309

G. 3 2 5 7 12 19

H. 8 1 9 10 19 29

I. 7 2 9 11 20 31

J. 4 2 6 8 14 22

K. 3 3 6 9 15 24

L. 5 10 15 25 40 65

M. 8 4 12 16 28 44

N. 0 7 7 14 21 35

O. 3 10 13 23 36 59

P. 5 4 9 13 22 35

Q. 6 7 13 20 33 53

R. 8 7 15 22 37 59

S. 10 20 30 50 80 130

T. 9 16 25 41 66 107

U. 13 8 21 29 50 79

V. 14 9 23 32 55 87

What Can You Say?

Carl had exactly as many stamps as Alice and Betty had together. Dan had exactly as many stamps as Carl and Betty had together.

Alice had 9 stamps. Dan had 4 less than 5 times that many.

Five times as many as Alice is 45, so Dan must have had 41.

If Betty had 15 stamps, we would know that Carl had 24. Then Dan would have as many as Betty and Carl put together, or 39 (which is not true). Betty, therefore, could not have had 15. Betty actually had 16, and Carl must have had 25.

(continued from previous page)

A. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

B. 0, 10, 20, 30, 40, 50, 60, 70, 80, or 90.

C. 0, 100, 200, 300, 400, . . . 800, or 900.

Remember: You must not use more than one number from a list in any one example.

1	2
25 x 10 = 250	17 x 20 = 340
25 x 7 = 175	17 x 5 = 85
25 x 17 = 425	17 x 25 = 425

3	4
9 x 100 = 900	13 x 200 = 2,600
9 x 40 = 360	13 x 60 = 780
9 x 8 = 72	13 x 7 = 91
9 x 148 = 1332	13 x 267 = 3,471

5	6
4 x 800 = 3,200	16 x 600 = 9,600
4 x 70 = 280	16 x 9 = 144
4 x 9 = 36	16 x 0 = 0
4 x 879 = 3516	16 x 609 = 9744

7	8
36 x 500 = 18,000	18 x 50 = 900
36 x 40 = 1,440	18 x 3 = 54
36 x 540 = 19,440	18 x 53 = 954

Bill's Plan

Bill told Mrs. Brown he used a plan in tackling these problems. She asked him to explain it to the class.

"Well, my first step is to look at the bottom line. For example in Exercise 8 it is:

18 x ? = 954

"Then I think about it this way:

18 x ? = 954

18 x 50 = 900

"and I look at the largest number in the lists for the blue block so the product won't be more than 954. In this case, it's 50. So I multiply 18 by 50 and subtract to find out how much of the 954 I still need to work on.

18 x ? = 954

− 18 x 50 = 900

54

"Now the problem is cut down in size. The rest of the work is much easier.

"Here are some examples for you to try — using the FIRST STEP of my plan."

A.

24 x 22 = 528

− 24 x 20 = 480

48

B.

15 x 17 = 255

− 15 x 10 = 150

105

(There are more examples of Bill's first step on the next page.)

Please do the examples below "in your head." Did you find it convenient to do them mentally?

	C.	D.	E.	F.
9	49 x 20 = 980	35 x 200 = 7,000	17 x 80 = 1,360	38 x 400 = 15,200
10	83 x 30 = 2,490	72 x 400 = 28,800	24 x 90 = 2,160	79 x 800 = 63,200

Notes on Pupil Page 92

Tasks: *In exercises 1-8 pupils use the new rules to make true sentences. In exercises 9 and 10 pupils perform mental calculations to determine the missing products. In exercises A and B pupils use the first step of Bill's plan to make true sentences.*

Purpose: To explore a rationale for a long division algorithm for whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: The game of Fewest Numbers continues to depend on lists which force every solution into the form of the usual algorithm.

Bill has a plan for solving such problems. His strategy for working out:

$$18 \times \_\_\_\_ = 954$$

is to find the biggest number in any of the lists which, when multiplied by 18, is not more than 954. This is, you will notice, precisely what we do as we begin the usual division algorithm:

$$18 \overline{)\,954}\quad 5\text{ ↙}$$

We place 5 in that spot because it will end up in the 10's place and represent 50. We select it because it is the largest multiple of 10 or 100 that, when multiplied by 18, is not more than 954. And when we proceed:

$$\begin{array}{r} 5 \\ 18 \overline{)\,954} \\ \underline{90} \\ 54 \end{array}$$

we are completing Bill's first step. If we've taken care of 18×50 out of the 954 stones we are dividing into 18 piles with the same number in each, then how many have we left? Bill will later extend his approach so that his full account of the procedure will look like this:

$$18 \times \underline{53} = 954$$

$$\begin{array}{r} 3 \\ 50 \\ 18 \overline{)\,954} \\ \underline{900} \\ 54 \\ \underline{54} \end{array}$$

Now we are merely a couple of details away from the most efficient way to do a long division computation.

Practice sheet 92 presents a kind of search—during which one must carry out many, many computations. Some may wonder if all whole numbers can be expressed as the difference of two square numbers. You might try 2 and 6 and 10—the answer in each case is No. Can you express all odd numbers as the difference of square numbers? (Don't tell, but the answer is Yes.)

PS-92

0 x 0 = 0	4 x 4 = 16	8 x 8 = 64	12 x 12 = 144
1 x 1 = 1	5 x 5 = 25	9 x 9 = 81	13 x 13 = 169
2 x 2 = 4	6 x 6 = 36	10 x 10 = 100	14 x 14 = 196
3 x 3 = 9	7 x 7 = 49	11 x 11 = 121	15 x 15 = 225

This is a list of the Square Numbers from 0 through 225:

(0) 1 4 9 16 25 36 49 64 81 100 121 144 169 196 225

Each of the numbers listed below is the difference of a pair of Square Numbers (excluding 0). Find a correct pair of Square Numbers for each of the listed differences.

3 = 4 − 1	*21 = 25 − 4	45 = 81 − 36	*72 = 81 − 9
5 = 9 − 4	23 = 144 − 121	45 = 49 − 4	*75 = 100 − 25
7 = 16 − 9	*24 = 49 − 25	15 = 64 − 49	77 = 81 − 4
8 = 9 − 1	25 = 169 − 144	51 = 100 − 49	84 = 100 − 16
9 = 25 − 16	*27 = 36 − 9	52 = 196 − 144	85 = 121 − 36
11 = 36 − 25	28 = 64 − 36	55 = 64 − 9	88 = 169 − 81
12 = 16 − 4	29 = 225 − 196	*56 = 81 − 25	91 = 100 − 9
13 = 49 − 36	*32 = 81 − 49	57 = 121 − 64	95 = 144 − 49
15 = 16 − 1	33 = 49 − 16	60 = 64 − 4	*96 = 100 − 4
16 = 25 − 9	35 = 36 − 1	*63 = 64 − 1	99 = 100 − 1
17 = 81 − 64	36 = 100 − 64	64 = 100 − 36	104 = 225 − 121
19 = 100 − 81	39 = 64 − 25	65 = 81 − 16	*105 = 121 − 16
20 = 36 − 16	*40 = 49 − 9	69 = 169 − 100	200 = 225 − 25

Use Square Numbers to make true statements

All blanks below are reserved for Square Numbers greater than zero selected from the list at the top of this page. Do not repeat the same number in any one sentence below. Make each sentence a true statement.

9 + 16 = 25	49 + 64 = 113
25 + 144 = 169	25 + 49 = 74
36 + 64 = 100	49 + 121 = 170

36 + 49 + 1 = 86

25 + 49 + 9 = 83

*36 + 64 + 49 = 149

9 + 81 + 100 = 190

*25 + 49 + 121 = 195

*49 + 64 + 81 = 194

*OTHER CORRECT ANSWERS POSSIBLE.

1

23 x ? = 345

23 x 10 = 230

115

2

72 x ? = 2,664

72 x 30 = 2160

504

3

105 x ? = 2,625

105 x 20 = 2,100

525

4

215 x ? = 3,225

215 x 10 = 2150

1075

5

87 x ? = 8,874

87 x 100 = 8700

174

6

39 x ? = 8,190

39 x 200 = 7800

390

7

48 x ? = 432

48 x 9 = 432

0

8

39 x ? = 1,404

39 x 30 = 1,170

234

Second Step in Bill's Plan

"After I've reduced the size of the problem I'm usually not done.

"So, I look at what's left in this way, using the last problem as an example:

39 x ? = 1,404

39 x 30 = 1,170

234

39 x 6 = 234

0

"Again, I look for the largest number in the lists for the second blue block so the product won't be more than 234. In this case it's 6. So I multiply 39 × 6 and subtract

39 x ? = 1,404

39 x 30 = 1,170

234

39 x 6 = 234

0

"This problem is finished and I know that

39 x 36 = 1,404

Here are exercises for you to solve using Step One and Step Two of Bill's plan:

9

15 x ? = 345

15 x 20 = 300

45

15 x 3 = 45

0

10

43 x ? = 4,558

43 x 100 = 4300

258

43 x 6 = 258

0

11

51 x ? = 19,380

51 x 300 = 15,300

4080

51 x 80 = 4080

0

12

125 x ? = 3,125

125 x 20 = 2,500

625

125 x 5 = 625

0

13

23 x ? = 3,965

23 x 100 = 2,300

1,656

23 x 70 = 1,610

46

14

17 x ? = 3,995

17 x 200 = 3400

595

17 x 30 = 510

85

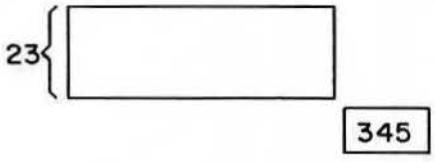
These last two exercises aren't completely solved yet . . . see the next page.

Tasks: In exercises 1-8 pupils use the first step of Bill's plan to make true sentences. In exercises 9-14 they use the first two steps of Bill's plan to make true sentences.

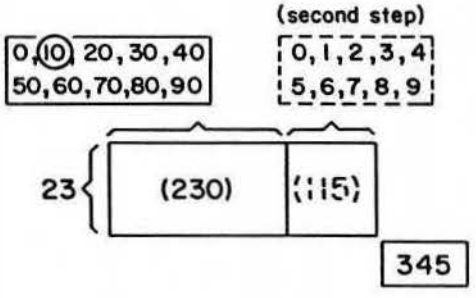
Purpose: To explore a rationale for a long division algorithm for whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: It will help pupils to understand Bill's plans if you draw a sketch or two. Example 1 on pupil page 93 can be understood as asking: If 345 tiles are arranged in 23 rows and each row has the same number of tiles, how many tiles are in each row?



In the first step of the solution, we determine the largest multiple of 10 per row.



If we tried 20, we would get 460 tiles, which is more than the total number of tiles—345. So we select 10, and this accounts for 23 rows of 10, or 230 tiles. This leaves $345 - 230 = 115$ tiles to consider. When the largest multiple of 10 has been determined we have taken a very large step toward the solution. We have cut our problem down in size.

Bill's second step, of course, would be to complete the job by finding how the 115 remaining tiles must be arranged in the 23 rows. He would find that 5 more in each of the 23 rows would take care of the 115 left over after the first step.

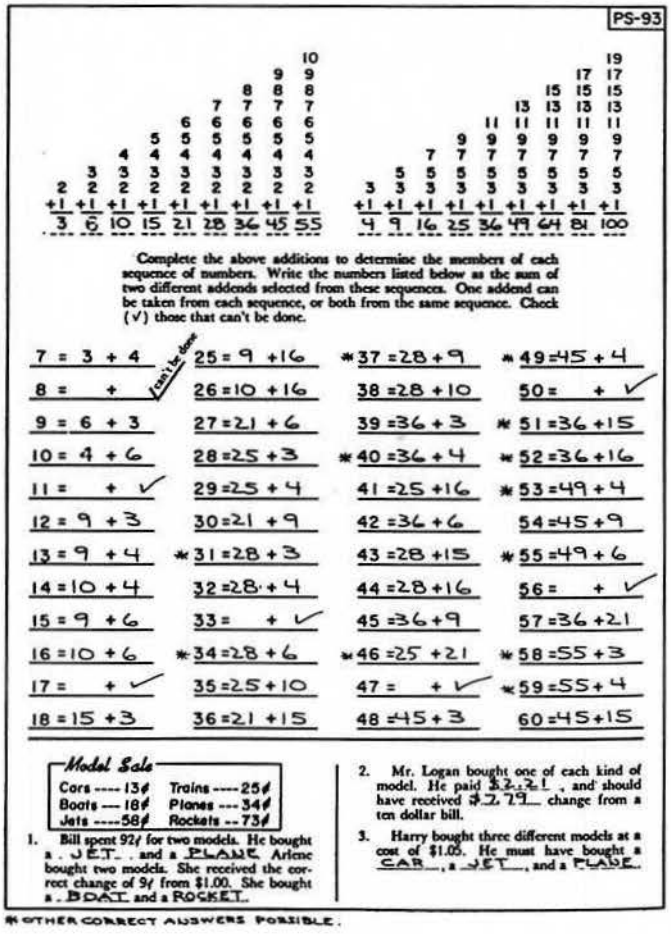
Notice that in example 7 all multiples of 10 (other than 0×10) multiplied by 48 are greater than 432, so the first nonzero selection must be made from the list of numbers less than 10—in this case, 9.

Eventually, pupils will not think about the details of such arguments or consider physical representations when they work—unless they run into problems that stump them.

On practice sheet 93 a problem is presented: How many numbers from 7 through 60 can you express as the sum of two different numbers selected from a list of triangular numbers (3, 6, 10, etc.) and a list of square numbers (4, 9, 16, etc.) greater than 1? If pupils claim that a number cannot be expressed in this way, you might ask them to give a convincing argument to support the claim. For example, to find two numbers whose sum is 11, we need only consider numbers less than 11 as candidates

candidates	10	9	6	4	3
needed to make 11	1	2	5	7	8

and we see that none of these pairs is in the lists.


PS-93

2 +1 3 | 3 +1 6 | 4 +1 10 | 5 +1 15 | 6 +1 21 | 7 +1 28 | 8 +1 36 | 9 +1 45 | 10 +1 55

3 +1 4 | 5 +1 9 | 7 +1 16 | 9 +1 25 | 11 +1 36 | 13 +1 49 | 15 +1 64 | 17 +1 81 | 19 +1 100

Complete the above additions to determine the members of each sequence of numbers. Write the numbers listed below as the sum of two different addends selected from these sequences. One addend can be taken from each sequence, or both from the same sequence. Check (✓) those that can't be done.

7 = 3 + 4	25 = 9 + 16	*37 = 28 + 9	*49 = 45 + 4
8 = + can't be done	26 = 10 + 16	38 = 28 + 10	50 = + ✓
9 = 6 + 3	27 = 21 + 6	39 = 36 + 3	*51 = 36 + 15
10 = 4 + 6	28 = 25 + 3	*40 = 36 + 4	*52 = 36 + 16
11 = + ✓	29 = 25 + 4	41 = 25 + 16	*53 = 49 + 4
12 = 9 + 3	30 = 21 + 9	42 = 36 + 6	54 = 45 + 9
13 = 9 + 4	*31 = 28 + 3	43 = 28 + 15	*55 = 49 + 6
14 = 10 + 4	32 = 28 + 4	44 = 28 + 16	56 = + ✓
15 = 9 + 6	33 = + ✓	45 = 36 + 9	57 = 36 + 21
16 = 10 + 6	*34 = 28 + 6	*46 = 25 + 21	*58 = 55 + 3
17 = + ✓	35 = 25 + 10	47 = + ✓	*59 = 55 + 4
18 = 15 + 3	36 = 21 + 15	48 = 45 + 3	60 = 45 + 15

Model Sale

Cars ---- 13¢	Trains ---- 25¢
Boats --- 18¢	Planes --- 34¢
Jets ---- 58¢	Rockets -- 73¢

1. Bill spent 92¢ for two models. He bought a JET and a PLANE. Arlene bought two models. She received the correct change of 9¢ from $1.00. She bought a BOAT and a ROCKET.

2. Mr. Logan bought one of each kind of model. He paid $2.31, and should have received $.79 change from a ten dollar bill.

3. Harry bought three different models at a cost of $1.05. He must have bought a CAR, a JET, and a PLANE.

*OTHER CORRECT ANSWERS POSSIBLE.

(continued from previous page using same 3 lists)

A. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

B. 0, 10, 20, 30, 40, 50, 60, 70, 80, or 90.

C. 0, 100, 200, 300, 400, . . . 800, or 900.

"The last two problems weren't completed after the first and second steps," Bill explained. "So, I simply add another step . . . and that is the end!" (Why?)

Here are some examples using Bill's 3-step plan:

1	2
9 x ? = 1,287	25 x ? = 8,175
9 x 100 = 900	25 x 300 = 7,500
387	675
9 x 40 = 360	25 x 20 = 500
27	175
9 x 3 = 27	25 x 7 = 175
0	0

3	4
38 x ? = 8,322	118 x ? = 18,998
38 x 200 = 7,600	118 x 100 = 11,800
722	7,198
38 x 10 = 380	118 x 60 = 7,080
342	118
38 x 9 = 342	118 x 1 = 118
0	0

Mrs. Brown helped Bill explain his plan to the class.

Soon, everyone was able to go through the steps until the problems were completed.

Then, Mrs. Brown told the class that this plan was very much like what she called a "kind of long division."

"The only difference," she said, "is that I keep track of my arithmetic in a little different way."

Bill's Plan | Mrs. Brown's Method

A

Bill's Plan	Mrs. Brown's Method
9 x ? = 1,287	9 x ? = 1,287
9 x 100 = 900	x 3
387	x 40
9 x 40 = 360	x 100
27	9 | 1,287
9 x 3 = 27	900
0	387
	360
	27
	27
	0

B

Bill's Plan	Mrs. Brown's Method
27 x ? = 7,101	27 x ? = 7,101
27 x 200 = 5,400	3
1,701	60
27 x 60 = 1,620	200
81	27 | 7,101
27 x 3 = 81	5400
0	1701
	1620
	81
	81
	0

94-95 Formalization of long division algorithm

Tasks: *In exercises 1-4 pupils use Bill's three-step plan to make true sentences. In exercises A and B they use both Bill's plan and Mrs. Brown's method to make true sentences.*

Purpose: To explore a rationale for a long division algorithm for whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: There is a rule of thumb that can be used to test for understanding. It is: Could you find the correct answer if you forgot the usual rule?

As computational techniques are developed carefully, our thought processes change from reasoning to rote. In the interests of convenience and speed we memorize a step-by-step procedure, confident it will produce correct results. If we lose confidence in that procedure or forget it, can we fall back to the reasoning stage in our development and work confidently? That is the true test for understanding.

An alternative road to computational facility is to begin and end with rote learning. The many steps in the long division process can, perhaps, be learned as a succession of rules: first you do this, next you do that, etc. Such a development requires new rules or novel variations on old ones for every new type of problem. And the final equipment—sufficient rules for all possible situations—requires a great amount of memorization. To remember the necessary rules and procedures over months of disuse is simply too much for most pupils. They would require constant refresher courses on which step follows which.

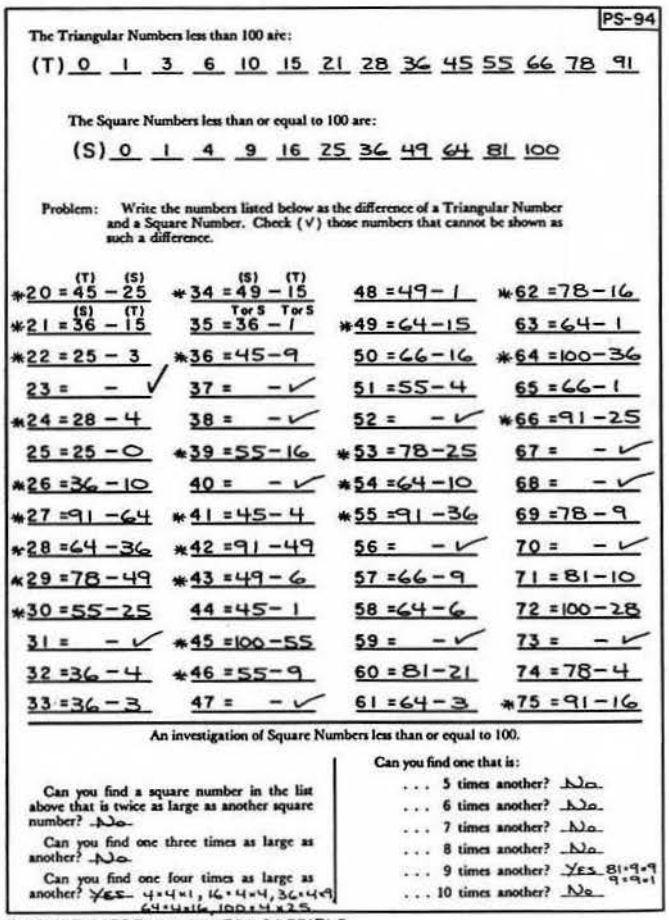
This equipment, this complicated set of steps learned by rote memorization, leaves a pupil a slave to his memory—leaves him no way out when his memory falters.

If asked what he would do if he forgot the rules, he would reveal his lack of understanding by his helplessness.

Pupil pages 94 and 95 conclude our development of long division a few very minor steps short of the traditional algorithm. We suspend the discussion at the point where each step is still recorded in a way that shows why it was taken and what it contributes to the ultimate solution.

On practice sheet 94 we pose the problem of expressing specified numbers as the difference of two numbers, one from each of the two given lists. This practice sheet can be used as a model for activities that require much mental arithmetic, simply by changing the lists of numbers.

The series of questions about square numbers at the bottom of the practice sheet introduces an idea that will be explored in depth much later. It is: Some numbers such as 2, 3, 5, 6, 7, 8, 10, etc., are not only not square numbers—they are not the square of any fraction. That is, these numbers (and many more) do not have square roots that are fractions.



PS-94

The Triangular Numbers less than 100 are:

(T) 0 1 3 6 10 15 21 28 36 45 55 66 78 91

The Square Numbers less than or equal to 100 are:

(S) 0 1 4 9 16 25 36 49 64 81 100

Problem: Write the numbers listed below as the difference of a Triangular Number and a Square Number. Check (✓) those numbers that cannot be shown as such a difference.

*20 = 45 − 25 (T) (S)	*34 = 49 − 15 (S) (T)	48 = 49 − 1	*62 = 78 − 16
*21 = 36 − 15 (S) (T)	35 = 36 − 1 T or S T or S	*49 = 64 − 15	63 = 64 − 1
*22 = 25 − 3	*36 = 45 − 9	50 = 66 − 16	*64 = 100 − 36
23 = − ✓	37 = − ✓	51 = 55 − 4	65 = 66 − 1
*24 = 28 − 4	38 = − ✓	52 = − ✓	*66 = 91 − 25
25 = 25 − 0	*39 = 55 − 16	*53 = 78 − 25	67 = − ✓
*26 = 36 − 10	40 = − ✓	*54 = 64 − 10	68 = − ✓
*27 = 91 − 64	*41 = 45 − 4	*55 = 91 − 36	69 = 78 − 9
*28 = 64 − 36	*42 = 91 − 49	56 = − ✓	70 = − ✓
*29 = 78 − 49	*43 = 49 − 6	57 = 66 − 9	71 = 81 − 10
*30 = 55 − 25	44 = 45 − 1	58 = 64 − 6	72 = 100 − 28
31 = − ✓	*45 = 100 − 55	59 = − ✓	73 = − ✓
32 = 36 − 4	*46 = 55 − 9	60 = 81 − 21	74 = 78 − 4
33 = 36 − 3	47 = − ✓	61 = 64 − 3	*75 = 91 − 16

An investigation of Square Numbers less than or equal to 100.

Can you find a square number in the list above that is twice as large as another square number? No.

Can you find one three times as large as another? No.

Can you find one four times as large as another? Yes 4·4=1, 16·4=4, 36·4=9, 64·4=16, 100·4=25

Can you find one that is:

... 5 times another? No.
... 6 times another? No.
... 7 times another? No.
... 8 times another? No.
... 9 times another? Yes 81·9=9, 9·9=1
... 10 times another? No.

* OTHER CORRECT ANSWERS POSSIBLE.

Try Bill's plan and Mrs. Brown's method on these examples:

C

32x 24 = 768

32x 20 = 640

128

32x 4 = 128

0

32x 24 = 768

```
      4
     20
32 | 768
     640
     128
     128
       0
```

D

16x 123 = 1,968

16x 100 = 1,600

368

16x 20 = 320

48

16x 3 = 48

0

16x 123 = 1,968

```
      3
     20
    100
16 | 1,968
     1600
      368
      320
       48
       48
        0
```

E

24x 309 = 7,416

24x 300 = 7,200

216

24x 9 = 216

0

24x 309 = 7,416

```
      9
    300
24 | 7,416
     7200
      216
      216
        0
```

F

39x 70 = 2,730

39x 70 = 2,730

```
     70
39 | 2,730
     2730
        0
```

"Let me show you a short cut in the method I've been using," said Mrs. Brown.

"I don't have to use several blue blocks as I have been doing. I notice that all the numbers in each list (except 0) have a first digit from 1 to 9 . . . and the rest of the digits are 0's. So, I usually use the following short cut:"

[1] [2]

48x 16 = 768

```
     16
48 | 768
     480
     288
     288
       0
```

35x 24 = 840

```
     24
35 | 840
     700
     140
     140
       0
```

"And I can leave out the green band since it simply restates the problem."

[3] [4]

```
     41
63 | 2,583
     2520
       63
       63
        0
```

```
     87
15 | 1,305
     1200
      105
      105
        0
```

[5] [6]

```
     142
35 | 4,970
     3500
     1470
     1400
       70
       70
        0
```

```
     207
43 | 8,901
     8600
      301
      301
        0
```

Tasks: In exercises C-F pupils use Bill's plan and Mrs. Brown's method to make true sentences. In exercises 1-6 they use Mrs. Brown's shortcut to complete the examples.

Purpose: To further explore a rationale for a long division algorithm for whole numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

The Lesson: We leave the long division algorithm at this point just a "few zeros away" from the traditional form. These zeros help point to the reasoning that makes the procedure logical and dependable.

If, at this point, any pupil is asked what he would do if he forgot the rules, he ought to be able to suggest an alternative course of action—less efficient, perhaps, but just as reasonable.

The work on practice sheets 95 and 96 suggests a bit of the arithmetic to come . . . a way of extending our ideas about numbers.

We have already considered the thermometer and the way we talk about "7 degrees below zero." It is quite reasonable that we should develop a shorthand to use in distinguishing between "8 degrees above zero" and "8 degrees below zero." One way would be to write:

$$-8° \text{ and } +8°$$

The new shorthand is certainly an easy one to remember. And we can easily adapt it to help us distinguish between "8 seconds before blast off" and "8 seconds after blast off":

$$-8 \text{ sec. and } +8 \text{ sec.}$$

If we lost \$8 in a transaction and then made an \$8 profit in another, we could report this as:

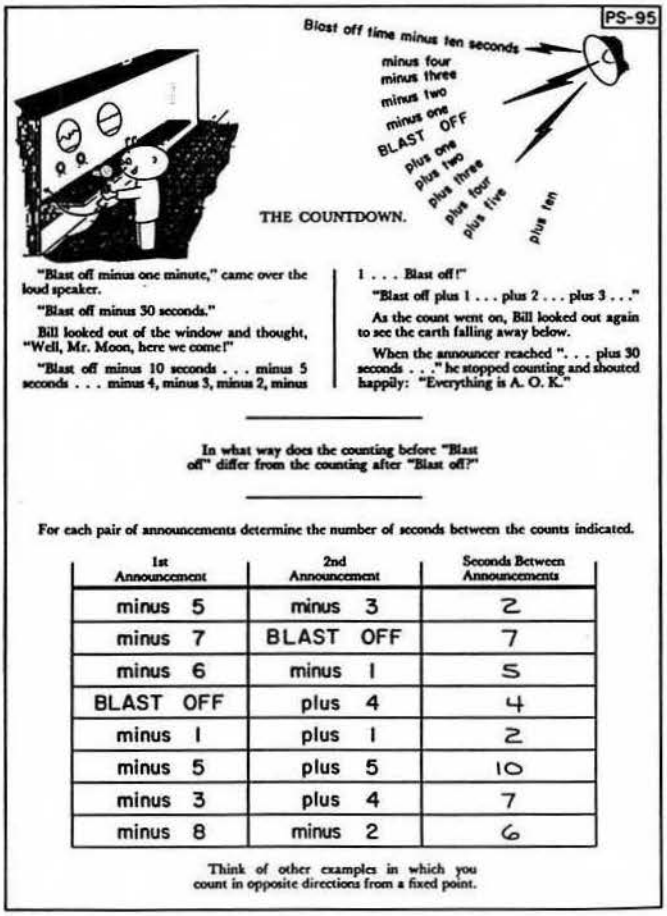
$$-8 \text{ dollars and } +8 \text{ dollars}$$

In each of these contexts, the number of degrees, seconds, and dollars is 8 and we use the "−" and "+" to distinguish two opposite directions—below and above zero degrees, before and after blastoff, loss and profit. Listed without any reference to a particular context:

$$-8 \text{ and } +8$$

are sometimes referred to as opposite numbers, as negative and positive numbers, as directed numbers, or as signed numbers.

On practice sheet 95 we consider time announcements during the launching of a rocket and note the time elapsed between them. On practice sheet 96 we consider opposite numbers as suggesting labels for the extended number line. The purpose in these practice sheets is to suggest some of the arithmetic to come that will expand our ability to record results of experiments or experiences in the physical world.



PS-95

THE COUNTDOWN.

"Blast off minus one minute," came over the loud speaker.

"Blast off minus 30 seconds."

Bill looked out of the window and thought, "Well, Mr. Moon, here we come!"

"Blast off minus 10 seconds . . . minus 5 seconds . . . minus 4, minus 3, minus 2, minus 1 . . . Blast off!"

"Blast off plus 1 . . . plus 2 . . . plus 3 . . ."

As the count went on, Bill looked out again to see the earth falling away below.

When the announcer reached ". . . plus 30 seconds . . ." he stopped counting and shouted happily: "Everything is A. O. K."

In what way does the counting before "Blast off" differ from the counting after "Blast off?"

For each pair of announcements determine the number of seconds between the counts indicated.

1st Announcement	2nd Announcement	Seconds Between Announcements
minus 5	minus 3	2
minus 7	BLAST OFF	7
minus 6	minus 1	5
BLAST OFF	plus 4	4
minus 1	plus 1	2
minus 5	plus 5	10
minus 3	plus 4	7
minus 8	minus 2	6

Think of other examples in which you count in opposite directions from a fixed point.

Following Directions

1. Multiples of 2 are: 0, 2, 4, 6, 8, 10, 12, 14, 16, etc.

2. Multiples of 3 are: 0, 3, 6, 9, 12, 15, 18, 21, 24, etc.

3. Multiples of 5 are: 0, 5, 10, 15, 20, 25, 30, 35, 40, etc.

4. Multiples of 7 are: 0, 7, 14, 21, 28, 35, 42, 49, 56, etc.

In the examples below, list 12 numbers that fit the descriptions given. Write the numbers in order — from the smallest to the largest.

5. Multiples of 2 that are also multiples of 3:

0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66

6. Multiples of 3 that are also multiples of 4:

0, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132

7. Multiples of 3 that are also multiples of 5:

0, 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165

8. Multiples of 2 that are neither multiples of 3 nor multiples of 5:

2, 4, 8, 14, 16, 22, 26, 28, 32, 34, 38, 44

9. Multiples of 3 that are neither multiples of 2 nor multiples of 5:

3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87

10. Multiples of 5 that are neither multiples of 2 nor multiples of 3:

5, 25, 35, 55, 65, 85, 95, 115, 125, 145, 155, 175

96-97 Following Directions: Achievement Tests

Tasks: Pupils follow the indicated directions.

Purpose: To explore the concepts of multiples and common multiples of whole numbers.

Unifying Ideas: Multiplication and division; functions and relations.

The Lesson: Pupil pages 96 and 97 are designed for use as a checkup. The questions on the latter are asked in a form used by most makers of standardized tests.

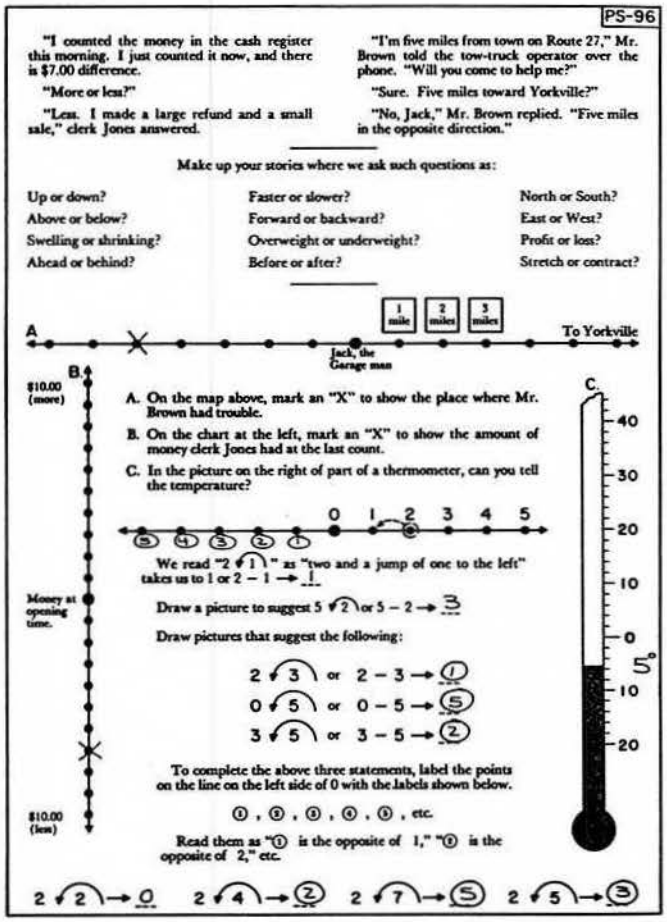
On practice sheet 96 the notion of the opposite of familiar numbers is used to help create labels for an extended number line—extended on the side of 0 that is in the opposite direction from the familiar labels.

Discuss situations like the following in which direction must be considered in order to report certain events more fully:

Alice noted that her weight had changed 2 pounds since school started. Her new weight is 58 pounds. What did she weigh when school started? (56 or 60 pounds)

Mr. Alexander said: I think we will arrive at Aunt Marie's at 3 o'clock. He missed his estimate by just half an hour. What time did the Alexander's arrive? (2:30 or 3:30)

A cricket started at 5 on the number line and took a jump of 3. Where did he land? (at 2 or at 8)


PS-96
"I counted the money in the cash register this morning. I just counted it now, and there is $7.00 difference.
"More or less?"
"Less. I made a large refund and a small sale," clerk Jones answered.
"I'm five miles from town on Route 27," Mr. Brown told the tow-truck operator over the phone. "Will you come to help me?"
"Sure. Five miles toward Yorkville?"
"No, Jack," Mr. Brown replied. "Five miles in the opposite direction."
Make up your stories where we ask such questions as:
Up or down?
Above or below?
Swelling or shrinking?
Ahead or behind?
Faster or slower?
Forward or backward?
Overweight or underweight?
Before or after?
North or South?
East or West?
Profit or loss?
Stretch or contract?
A
1 mile
2 miles
3 miles
To Yorkville
Jack, the Garage man
B
$10.00 (more)
Money at opening time.
$10.00 (less)
C
A. On the map above, mark an "X" to show the place where Mr. Brown had trouble.
B. On the chart at the left, mark an "X" to show the amount of money clerk Jones had at the last count.
C. In the picture on the right of part of a thermometer, can you tell the temperature?
0 1 2 3 4 5
We read "2 1" as "two and a jump of one to the left" takes us to 1 or 2 − 1 → 1
Draw a picture to suggest 5 2 or 5 − 2 → 3
Draw pictures that suggest the following:
2 3 or 2 − 3 → 1
0 5 or 0 − 5 → 5
3 5 or 3 − 5 → 2
To complete the above three statements, label the points on the line on the left side of 0 with the labels shown below.
①, ②, ③, ④, ⑤, etc.
Read them as "① is the opposite of 1," "② is the opposite of 2," etc.
40
30
20
10
0
10
20
5°
2 2 → 0
2 4 → 2
2 7 → 5
2 5 → 3

Following Directions

Mark the correct answer. Mark the furthest from correct answer.

How many days are there in a week? (1) 4 (2) 7 (3) 28 (4) 30

(1) (2) (3) (4) 23 ← The difference between the answers you marked.

1

Mary can stay three quarters of an hour. How many minutes are there in three quarters of an hour?

(1) 40 (2) 45 (3) 25 (4) 50

(1) (2) (3) (4) 20

2

Apples are 7 for 25¢. How many could you buy for $3.00?

(1) 21 (2) 12 (3) 73 (4) 84

(1) (2) (3) (4) 72

3

Children's tickets are 45¢ and adults' tickets are 80¢. How much will 3 adult and 4 children's tickets cost?

(1) $4.20 (2) $2.50 (3) $4.55 (4) $1.25

(1) (2) (3) (4) $2.95

4

Harry bought a model boat for 67¢ and a model plane for 88¢. He gave the clerk a 5-dollar bill. How much change should he get back?

(1) $2.75 (2) $1.55 (3) $2.45 (4) $3.45

(1) (2) (3) (4) $1.90

5

"We've gone one fourth of the way to Aunt Helen's," Mr. Howard said. "We've driven 70 miles. How many miles is it from home to Aunt Helen's house?"

(1) 110 (2) 280 (3) 350 (4) 265

(1) (2) (3) (4) 170

6

The play starts at 7:15 P.M. We need 15 minutes to pick up Jane, 55 minutes to travel, and 10 minutes to park and walk to the theatre. When should we leave?

(1) 6:35 (2) 4:50 (3) 4:30 (4) 5:55

(1) (2) (3) (4) 1 hr., 25 min.

7

How many quarts will we need for 28 people if we need 2 quarts for each group of 7 people?

(1) 8 (2) 4 (3) 14 (4) 35

(1) (2) (3) (4) 27

8

On a map there is a note:

"Scale: 1 inch = 6 miles"

A length of $2\frac{1}{2}$ inches on the map represents how many miles?

(1) 12 (2) 15 (3) 19 (4) $8\frac{1}{2}$

(1) (2) (3) (4) $6\frac{1}{2}$

9

Mrs. Jones bought 6 pounds of fruit. Her correct change from a dollar bill was 46¢. What was the per pound cost of the fruit purchased?

(1) 6¢ (2) 40¢ (3) 9¢ (4) 15¢

(1) (2) (3) (4) 31¢

Notes on Pupil Page 97

Tasks: Pupils follow the indicated directions.

Purpose: To assess pupil achievement in solving word problems.

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

The Lesson: Pupil page 97 is the second part of a two-page checkup. You may notice that, in some of the questions on this pupil page, the level of difficulty is comparable to that which one encounters in standardized tests at the sixth-grade and seventh-grade level.

Letter Patterns

It was a rainy Saturday afternoon.

"Let's hunt for patterns in the letters of words," Numa suggested.

"A pattern in letters of words?"

"Yes, Numo. I pick a Pattern Indicator:

I **a b b c**

That means I want to find 4-letter words with the two middle letters alike," Numa explained.

"Fine," Numo agreed. "You keep a list as we think of them."

meet	room	seed
boot	ills	pool
feet	peep	eddy
		etc.

(Please fill in the blanks with your own words that fit the Pattern Indicator.)

"Numa, if I can use names, then 'Anne' would fit the Pattern Indicator."

"Fine, Numo, and you can use plurals, too — 'inn' wouldn't fit, but 'inns' fits the Pattern Indicator."

"How about 'deed?' The middle letters are alike, but the first is the same as the last."

"That's all right, too! Pattern Indicators only tell you the number of letters and which ones must be alike.

"Let's change the Pattern Indicator to:

II **r s t r**

Four-letter words with the same first and last letters."

that	lull	sits
else	pump	peep
tint	says	Anna
		etc.

"I can make the hunt more difficult," Numa told her brother, "if I combine the Pattern Indicators. We already have one. Can we find two more?"

III **z m m z**

deed	noon	boob
		peep
		toot
		sees
		etc.

They succeeded, but it was a long search.

"Let's try an easier one. My Pattern Indicator is:

IV **a b c c**

"Let's see if we can find 15 examples."

tree	miss	lull
muff	will, doll, purr	
buzz	ball, bull, tall	
call, free, mass, huff		
gull, glee, mess, hall		
		etc.

"That was easy. Next, let's find 5-letter words in which the last two letters are the same.

V **a b c d d**

three	bless	grass
class	shall	dress
stiff	gruff	trill
igloo	skull	cross
		etc.

"Next, try 5-letter words with the third and fourth letters alike."

VI **g r s s t**

sleep	broom	merry
pills	trees	rally
speed	stoop	floor
gully	happy	shoot
		etc.

98-99 Letter Patterns: a Diversion

Tasks: Pupils use the pattern indicators as guides to constructing words.

Purpose: To explore word patterns.

Unifying Ideas: Structure; functions and relations.

The Lesson: Pupil pages 98 and 99 might be considered as a bit of a diversion. However, (1) they are concerned with patterns, and mathematics might well be described as the study of patterns, and (2) they focus on a rule that has a most important counterpart in algebra.

The discussion begins with the pattern *abbc*. Clearly this calls for a four letter word in which the second and third letters are the same. The words *deed, noon, toot,* etc., meet the requirements—and in fact exceed them, since the pattern indicator requires only the second and third letters to be alike, not the first and fourth as well. In other words, in the pattern indicator *abbc*, the letters *a* and *c* are *permitted* to be different, but are not *required* to be different. Similarly, in algebra, we say that if

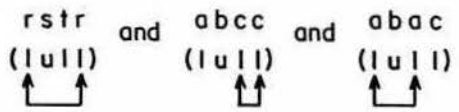
$$x+x=10$$

then $x=5$. However, we say that if

$$x+y=10$$

then any two numbers whose sum is 10 are solutions, including 5 and 5. Thus, if the letters are alike, they hold places for the same number. If they are different, they need not but may hold places for the same number.

Notice that the word *lull* is an example of the pattern *abaa*—and of *rstr*, *abcc*, and *abac* as well:



Every four-letter word, of course, can qualify as an example of the pattern indicator *abcd*.

The important idea is that this rule was made in an arbitrary fashion by the inventor of the game. Players might question the good sense of the inventor, but that is another question. As long as the rules are in effect, their strict interpretation will help players understand the game.

Numo and Numa made up a game for their friends. They wrote out a list of Pattern Indicators and a list of words.

Pattern Indicators

(1) abac	(2) fqqfh
(3) ghh	(4) abcab
(5) qrsrt	(6) wxyzxyz
(7) abbea	(8) abccda
(9) fgmqqmf	(10) abacad
(11) cdcecfc	(12) abcbda
(13) wxyzxw	(14) abbcb
(15) qrstqr	(16) xxyzy
(17) abcdceab	(18) abcaad
(19) pqqprspt	(20) xyxx

Players take turns. From the list below, each person tries to find a word that fits a Pattern Indicator in the list above. He writes the number of the Pattern Indicator next to the word and scores a point. If he can't find one, he loses his turn.

(7) seeds	(5) seven
(1) none	(18) fluffy
(20) lull	(8) spools
(13) serves	(2) tooth
(4) sense	(17) headache
(10) eleven	(3) too
(16) llama	(14) geese
(12) states	(11) Alabama
(19) Illinois	(9) dressed
(15) church	(6) singing

Did you match the Pattern Indicators with the words?

Maybe you can do as Numa and Numo did. Make up your own Pattern Indicators and words to fit them.

Here's another way to play a game of Pattern Indicators and words. A player makes up a Pattern Indicator based on a word he knows. The other player tries to find a word to fit the Pattern Indicator. If he does, he gets a point. If he can't, he says, "challenge." Then the one who wrote the Pattern Indicator must give a word that fits. If he can't, he loses a point.

Practice in the examples below. Find words to fit the Pattern Indicators.

1.
A abca
B tent

2.
B pqprst
A sister

3.
A wxyxz
B never fatal seven

4.
B amar
A away nine

5.
prqqp
sells

6.
abcdded
classes glasses

7.
abcddef
smaller

8.
axdrengq
any 8 letter word

9.
abccae
little

10.
pdpqbbr
fifteen

11.
abcdac
anyway

12.
abacdcca
nineteen

Notes on Pupil Page 99

Tasks: Pupils use the pattern indicators as guides to construct words.

Purpose: To explore word patterns.

Unifying Ideas: Structure; functions and relations.

The Lesson: Another version of the game of letter patterns follows from adding another rule to those already in effect; namely, vowels must replace vowels in the pattern and consonants must replace consonants. For example, for the pattern *bcaad*, the words *trees*, *broom*, and *steep* fit, while *pulls*, *merry*, and *rally* do not.

Pupils often wish to make up their own rules for the game of letter patterns. When they are in the role of inventor, they ought to be given complete freedom to change all or any of the rules we have been using. As the rules change, the game changes. This is an important idea in mathematics.

surprise!

I. THINK-OF-A-NUMBER!

Add 3 to it.

Double your result.

Subtract 4.

Divide your result by 2.

Subtract the number you thought of first.

Ahah! Your result is 1.

Try this with your friends.

If they get a final result other than 1, tell them that they must have made a mistake.

Will that surprise them!

You will learn to make up your own "think of a number" games.

II. Here is another "think of a number" game. Try four different examples.

	Examples: A	B	C	* D
Think of a number.	5	8	100	29
Add 4.	9	12	104	33
Multiply by 3.	27	36	312	99
Subtract the number you thought of first.	22	28	212	70
Divide by 2.	11	14	106	35
Subtract 2.	9	12	104	33
Subtract the number you thought of first.	4	4	4	4

*OTHER POSSIBILITIES.

III. Complete the columns below:

a.	1 x 9 = 09	0 + 9 = 9
b.	2 x 9 = 18	1 + 8 = 9
c.	3 x 9 = 27	2 + 7 = 9
d.	4 x 9 = 36	3 + 6 = 9
e.	5 x 9 = 45	4 + 5 = 9
f.	6 x 9 = 54	5 + 4 = 9
g.	7 x 9 = 63	6 + 3 = 9
h.	8 x 9 = 72	7 + 2 = 9
i.	9 x 9 = 81	8 + 1 = 9
j.	10 x 9 = 90	9 + 0 = 9

IV.

Do the multiplications before doing the subtractions (as in the first example below).

a.	1 x 2 - 0 x 3 = 2 - 0 = 2
b.	2 x 3 - 1 x 4 = 6 - 4 = 2
c.	3 x 4 - 2 x 5 = 12 - 10 = 2
d.	4 x 5 - 3 x 6 = 20 - 18 = 2
e.	5 x 6 - 4 x 7 = 30 - 28 = 2
f.	6 x 7 - 5 x 8 = 42 - 40 = 2
g.	7 x 8 - 6 x 9 = 56 - 54 = 2
h.	8 x 9 - 7 x 10 = 72 - 70 = 2
i.	9 x 10 - 8 x 11 = 90 - 88 = 2

Tasks: Pupils study the Think-of-a-Number game in I and complete four examples of another game in II. They complete the tables in III and IV, looking for patterns or surprises.

Purpose: To help pupils explore patterns in arithmetic through keeping records in tables.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Surprises I and II. At first, let each pupil study the *Think-of-a-Number* game in I. Then, in columns A, B, C, D of II the pupils are to complete the steps of a second *Think-of-a-Number* game. In column D, the pupils begin with any number they choose. If you need to make the problem somewhat simpler to get your class started, play the game on the board with 0, 1, 2, 3, and 4 in order. Guide pupils to make the following observations:

		Observation
Think of a number	0 1 2 3 4	numbers increase by steps of 1
Add 4	4 5 6 7 8	numbers increase by steps of 1
Multiply by 3	12 15 18 21 24	numbers increase by steps of 3
Take away number first thought of	12 14 16 18 20	numbers increase by steps of 2
Divide by 2	6 7 8 9 10	numbers increase by steps of 1
Take away 2	4 5 6 7 8	numbers increase by steps of 1
Take away number first thought of	4 4 4 4 4	all numbers the same.

Surprise III. In exercise III pupils complete a "9-times" table, writing the digits representing tens and units in different colored boxes. They then add the digits. If they wonder whether the pattern will continue, have them investigate for themselves. The sum of the digits will always be 9 or a multiple of 9.

Surprise IV. In exercise IV pupils will be surprised that the answer is always 2. They should be helped to notice that four consecutive numbers are being worked on in each example: 0, 1, 2, 3 in *a*; 1, 2, 3, 4 in *b*; 8, 9, 10, 11 in *f*, etc.

0, 1, 2, 3 $\quad (1 \times 2) - (0 \times 3) = 2$

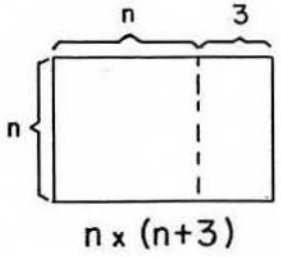
1, 2, 3, 4 $\quad (2 \times 3) - (1 \times 4) = 2$

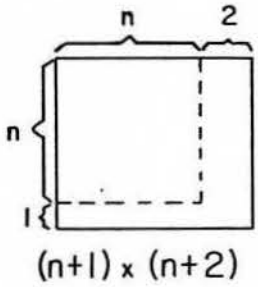
8, 9, 10, 11 $\quad (9 \times 10) - (8 \times 11) = 2$

The difference between the products of these pairs is always 2. Ask your class what they think $(42 \times 43) - (41 \times 44)$ would be. Have them work it out to check their guess. Next, have them complete examples like these:

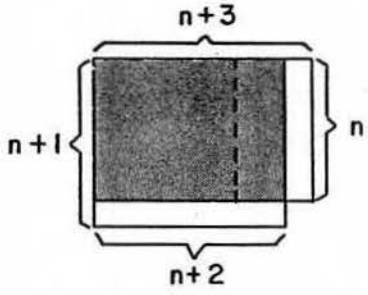
$$(\_\_\_ \times \_\_\_) - (100 \times 103) = 2$$
$$(85 \times 86) - (\_\_\_ \times \_\_\_) = 2$$

We do not explain to the pupils (and we doubt you should) why the difference is always 2 (although a similar "surprise" is explored on pupil pages 112-113). Here is why it happens. Let four consecutive whole numbers be n, $n+1$, $n+2$, and $n+3$. We can use rectangular regions to compare the sizes of $n \times (n+3)$ and $(n+1) \times (n+2)$. (The size of $n \times n$ is indicated by the dotted lines.)

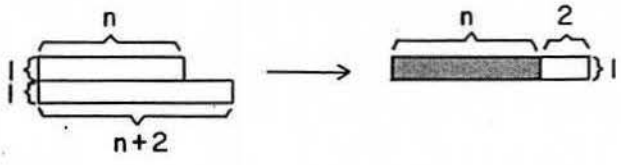




If we place one on top of the other,



we see that the shaded part can be eliminated from consideration, leaving only the two unshaded strips to compare.



We see that their difference (the unshaded part) is 2×1, or 2. This will be true for any whole number n.

V.

1	2	3	4	5	6	7
x 1	x 2	x 3	x 4	x 5	x 6	x 7
1	4	9	16	25	36	49
3	5	7	9	11	13	
2	2	2	2	2		

Carry out the multiplications. Record results in the grey spaces.

In each blue space, record the difference between the two numbers written above the blue space.

VI.

2	3	4	5	6	7	8
x 0	x 1	x 2	x 3	x 4	x 5	x 6
0	3	8	15	24	35	48
3	5	7	9	11	13	
2	2	2	2	2		

Use the green spaces similarly. In each, record the difference of the numbers written in the blue spaces above the green space.

Compare the results.

Did you expect what you see?

VII.

a. (0 x 3) ÷ 2 = 0

b. (1 x 4) ÷ 2 = 2

c. (2 x 5) ÷ 2 = 5

d. (3 x 6) ÷ 2 = 9

e. (4 x 7) ÷ 2 = 14

f. (5 x 8) ÷ 2 = 20

g. (6 x 9) ÷ 2 = 27

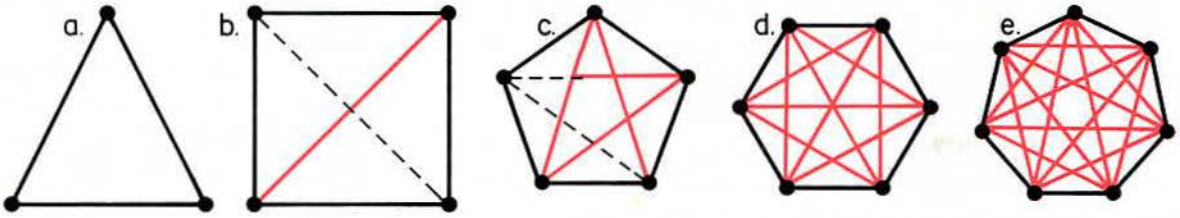
h. (7 x 10) ÷ 2 = 35

Find the results in the group of examples on the left. Record these results in the grey spaces.

a.	b.	c.	d.	e.	f.	g.	h.
0	2	5	9	14	20	27	35
2	3	4	5	6	7	8	
1	1	1	1	1	1		

In the blue spaces, record the differences of the numbers in the grey spaces. In the green spaces, record the differences of the numbers in the blue spaces.

VIII. Draw all the diagonals you can connecting the dots in each of the figures below. Record the number of diagonals in the grey spaces.



	a.	b.	c.	d.	e.
Number of diagonals →	0	2	5	9	14
Difference of numbers above →		2	3	4	5
Difference of numbers above →			1	1	1

Have you seen a similar result before?

Notes on Pupil Page 101

Tasks: Pupils complete four examples—V, VI, VII, VIII—looking for patterns or "surprises." In example VIII pupils use a ruler or straightedge to draw all diagonals in the regular geometric shapes.

Purpose: To help pupils explore patterns in arithmetic through making records. To provide practice in the use of a straightedge. To introduce the geometric notion of diagonal.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations; geometry.

The Lesson: Surprise V. On pupil page 101 the first example explores the facts that the difference of any two consecutive square numbers is an odd number and that if each of these differences is subtracted from the next, the result is 2.

Surprise VI. In the second example, the numbers 0, 1, 2, 3, etc., are taken in order and each is multiplied by 2 more than itself. The answers are 0, 3, 8, 15, 24, 35, 48. If a scientist found this sequence of numbers in the records of his experiments but did not know how it had been formed, he might wonder whether there was some simple rule he could use to discover the next number. Making a table of the differences between succeeding numbers, he would find:

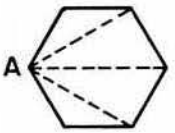
0	3	8	15	24	35	48
3	5	7	9	11	13	
2	2	2	2	2		

He would still not know how the sequence had been formed, but he could now predict that the next number in the middle row would be 15 and that the next number in the top row would be 63.

Surprise VII. In the third example each number is multiplied by 3 more than itself, and the result is divided by 2. To test the resulting numbers, 0, 2, 5, 9, 14, 20, 27, 35, a table of differences is made.

Surprise VIII. There is an interesting reason for choosing this particular rule. The numbers 0, 2, 5, 9, 14, etc., can be related to the number of diagonals of a triangle, quadrilateral, pentagon, hexagon, heptagon, etc. By drawing the diagonals of these figures and counting them, pupils are able to discover this relationship.

Review the idea of a diagonal line by drawing several geometric figures on the board. Pupils should note that a diagonal connects any two vertices that are not on the same straight line.



Each of the broken lines drawn from vertex A in the hexagon above are diagonals. Pupils should use a ruler or straightedge for this exercise.

The front runners will be able to predict how many diagonals there are for polygons with eight, nine, and ten sides without having to draw them. They can do so simply by extending the numerical pattern that they have recorded. Thus:

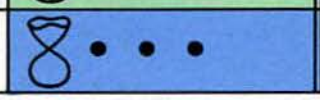
No. of sides	6	7	8	9	10	11	...
No. of diagonals	9	14	20	27	35		
		5	6	7	8		
			1	1	1	1	

This is, of course, the same pattern that appears in Surprise VII.

Anne gave each member of the class a bag and some counters. She said, "Think of a number. Put that number of counters in the bag, and tie it up.

Put 3 counters next to the bag. Now, if you tell me how many counters you have altogether, I will tell you what's in the bag."

Game I

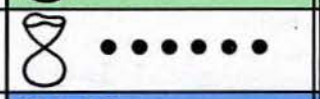
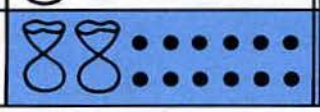
		Betty	Jim	Ruth	Ted	Ben
Think of a number. Put that many counters in the bag.		5	3	9	14	0
Put 3 counters next to the bag. How many counters altogether?		8	6	12	17	3

What numbers did Betty, Jim, Ruth, Ted, and Ben think of ?

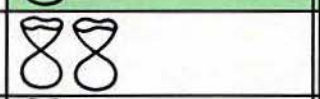
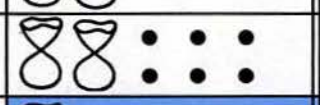
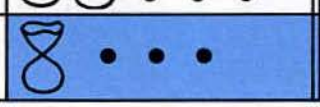
"That's too easy," Ben complained.

"All right," Anne continued, "you will each need extra bags and plenty of counters for the next games."

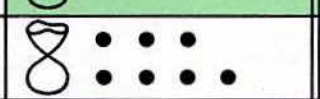
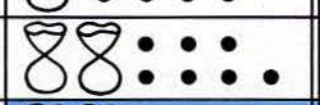
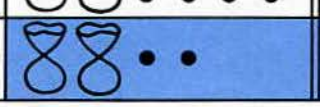
Game II

		Betty	Jim	Ruth	Ted	Ben
Think of a number. Put that many counters in the bag.		4	11	15	17	0
Add 6. You now have the bag and 6 extra counters.		10	17	21	23	6
Double it. You now have two bags and 12 extra counters. How many counters altogether?		20	34	42	46	12

Game III

Think of a number.		12	4	0	17	1
Double it.		24	8	0	34	2
Add 6.		30	14	6	40	8
Divide by 2. How many counters have you altogether?		15	7	3	20	4

Game IV

Think of a number.		7	10	25	0	1
Add 7.		14	17	32	7	8
Add the number you thought of first.		21	27	57	7	9
Subtract 5. How many have you altogether?		16	22	52	2	4

Tasks: Pupils complete four Think-of-a-Number *games involving bags and counters.*

Purpose: To help pupils reason through the steps of *Think-of-a-Number* games by using a physical representation of the development—bags or boxes and counters.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The bags and counters introduced on this page will be used on pupil page 103 to explain and invent tricks. The bag, containing an unknown number of counters, provides students with a way of visualizing the number someone has thought of. This helps them to reason about it. Game I is so simple that students do not really need the picture. The answer can be found by working backwards.

In Game II the students may have different ideas about what is meant by *double it.* Some may think: When you double 1 bag and 6 counters, you get 2 bags and 6 counters. Others may think that the result is 1 bag and 12 counters. In each case, the pupil is doubling only a part of the quantity. After Betty added 6 to her number there were 4 counters in the bag and 6 beside the bag, a total of 4+6. To double this she did not say 8+6 or 4+12. She said 8+12.

In Game III, students may reason: Betty's answer is 15, so she must have had 30 before she divided by 2, 24 before she added 6; and before she doubled she must have had 12. But by using the picture it is possible to reason: Her answer was 15; a bag and 3 counters makes 15 counters. Therefore, the bag must contain 12 counters; 12 was her number. This is much quicker—you can move directly from the record in blue to the record in green.

Game IV cannot be done at all by working backwards one step at a time, but it is easy if we use the pictures. Pupils can argue: Betty finished with 16. So two bags and two counters are 16 counters. Two bags must hold 14 counters; one bag must hold 7. That was Betty's number.

At this point you may wish to have pupils reconstruct records of *Think-of-a-Number* games, filling in missing statements and pictures. This will help them to understand better the correspondence between the words and the representations. Draw appropriate sketches:

Think-of-a-Number	
Add 2	
Triple it	
Subtract 6	
Double it	
Divide by 3	

Write appropriate directions:

Think-of-a-Number	

Fill in missing statements and sketches:

Think-of-a-Number	
Double it	
Subtract the number you first thought of	

"That game," Ben said, "reminds me of the trick I saw a little while ago. I can make these tricks up myself now. Here is an example:

Think of a number.
Double it. Then add 6.
Divide by 2.
Take away the number you first thought of."

YOUR RESULT IS 3!

Ben explained it this way:

Trick I

Think of a number.	Think of a number. Put that many counters in a bag.
Double it.	Double it. Now you have 2 bags with as many counters in each as the number you thought of first.
Add 6.	Add 6. Now you have 2 bags plus 6 extra counters.
Divide by 2.	Divide by 2. You remove 1 of the bags and 3 of the counters, leaving a bag and 3 counters.
Subtract the number you thought of first.	Now, subtract the number you thought of first.

Your result is 3!

Your secret number disappears from the scene, no matter how many counters were in the bag.

"Here are two more tricks," Ben volunteered. "They are longer, but they are based on the same idea."

The numbers Anne and Bill thought of are shown.
Also try a number you think of yourself.

Trick II

	Anne	Bill	You
Think of a number.	2	8	*
Add 3.	5	11	
Triple the result.	15	33	
Subtract 1.	14	32	
Subtract the number you thought of first.	12	24	
Divide by 2.	6	12	
Subtract the number you thought of first.	4	4	4

Trick III

	Anne	Bill	You
Think of a number.	10	5	*
Double it.	20	10	
Double it again.	40	20	
Subtract the number you thought of first.	30	15	
Add 6.	36	21	
Divide by 3.	12	7	
Subtract the number you thought of first.	2	2	2

* MANY POSSIBILITIES.

Notes on Pupil Page 103

Tasks: *Pupils study one* Think-of-a-Number *trick and complete two more.*

Purpose: To show pupils how to construct the tricks.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: This pupil page makes use of the bag and counter idea presented on pupil page 102. Trick I is simply Game III of pupil page 102, with a final step added. No new principle is involved in Tricks II and III.

The students will probably enjoy making up their own tricks. A trick can begin in any way whatever. It is always possible to find an ending that will make it work. The teacher may suggest a beginning: Think of a number. Add 1. Double. Subtract 1. Multiply by 3. There are 6 bags and 3 counters at this stage of the trick. The teacher may then challenge the class to finish it. A streamlined ending would be:

1. Add 9
2. Divide by 6
3. Subtract the number first thought of

The answer is 2. This can be checked by drawing bags and counters.

Pupils rarely have sufficient experience to see this quick way home. They may give a much longer ending, with several subtractions and divisions. This does not matter; it may even make the trick more impressive and give more practice in arithmetic. Pupils might present such endings as these, beginning at the stage with 6 bags and 3 counters:

1. Divide by 3
2. Subtract 1
3. Divide by 2
4. Add 5
5. Subtract your original number

Your number is 5

1. Double
2. Divide by 3
3. Add 10
4. Divide by 4
5. Add your original number
6. Subtract 1
7. Divide by 2
8. Subtract 1

Your number is 0

Tricks should be invented by drawing bags and counters, and checked by trying particular numbers. A trick has reached an ending only when there are no bags in the drawing.

Next day, Jerry said that he had made up tricks like Ben's but with larger numbers. Instead of drawing counters, Jerry writes numbers to keep track of the extra counters.

(Below is the number Ben thought of. Try a number you think of.)

Trick IV

		Ben	You
Think of a number.	8	19	*
Add 50.	8 + 50	69	
Double it.	88 + 100	138	
Add 48.	88 + 148	186	
Divide by 2.	8 + 74	93	
Subtract the number you thought of first.		74	

"Now," Jerry went on, "For the next trick I'm going to have to draw large numbers of bags unless we can agree that all I need to do is draw one bag and tell you how many I mean to draw."

Trick V

		Ben	You
Think of a number.	8	7	*
Multiply by 10.	10 8	70	
Add 130.	10 8 + 130	200	
Divide by 5.	2 8 + 26	40	
Divide by 2.	8 + 13	20	
Subtract the number you thought of first.		13	

"Those bags are hard to draw," Jerry complained. "So I'm not going to draw the top and bottom of the bag. All I shall draw is an X which looks very much like the letter *x*. So, for example, I write 3X + 6 when I mean 3 bags and 6 extra counters."

Trick VI

	Shorthand	Ben	You
Think of a number.	X	3	*
Add 47.	X + 47	50	
Double it.	2X + 94	100	
Subtract 75.	2X + 19	25	
Triple it.	6X + 57	75	
Subtract the number you thought of first.	5X + 57	72	
Add 18.	5X + 75	90	
Divide by 5.	X + 15	18	
Subtract the number you thought of first.		15	

Finish the picture and use Jerry's shorthand to show the following trick:

Trick VII

Words	Pictures	Shorthand
Think of a number.	8	X
Add 4.	8 • • • •	X + 4
Double it.	8 8 : :	2X + 8
Subtract 2.	8 8 : :	2X + 6
Add the number you thought of first.	8 8 8 : :	3X + 6
Divide by 3.	8 • •	X + 2
Subtract the number you thought of first.	• •	2

* MANY POSSIBILITIES.

Tasks: *Pupils complete four* Think-of-a-Number *tricks. In VII they draw pictures and complete algebraic shorthand representing the steps. In tricks IV-VI, each pupil makes up one example of his own.*

Purpose: To teach pupils algebraic shorthand to replace the bag symbol. To provide practice in reading and understanding algebraic expressions and to demonstrate the usefulness of such expressions.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: There is no reason why a trick should not begin: Think of a number. Multiply by 1000. Add 300. It would be wearisome, actually, to draw a thousand bags and three hundred counters. It is better to represent this by writing:

1000 ☒ + 300

All that the first column of this pupil page (tricks IV and V) is intended to do is to establish this simple shorthand.

Tricks VI and VII introduce a very small change. We erase the top and bottom of the bag. This is simply a sneaky way of bringing in x and eliminating the rather primary-looking bags.

We try to keep the old meanings with the new sign. On one occasion we may read $2x+6$ just as it is written, "two x plus six." On another occasion, we may read it as "two bags and six counters." Or again, we may read it as "twice the number thought of, with six added."

Have pupils read or write expressions like $3x+10$ in these three ways:

three $x+10$

three bags and ten counters

three times the number thought of, with ten added

Fred has an IDEA.

"If you show me the result at any stage of the games or tricks, I'll tell you the number the player was thinking of to start with. The only exceptions would be the last lines of the tricks, because there has to be at least one bag left in the picture."

Fred did it. Can you discover his method?

Game II — page 102

		Betty	Jim	Ruth	Ted	Ben
Think of a number.	8	4	11	15	17	0
2nd line →	8 ●●●●●●	10	17	21	23	6

Game III — page 102

Think of a number.	8	12	4	0	17	1
3rd line →	8 8 ●●● ●●●	30	14	6	40	8

Game IV — page 102

Think of a number.	8	7	10	25	0	1
4th line →	8 8 ●●	16	22	52	2	4

Trick II — page 103

		Anne	Bill
Think of a number.	8	2	8
5th line	8 8 ●●●● ●●●●	12	24

Trick III — page 103

		Anne	Bill
Think of a number.	8	10	5
5th line	8 8 8 ●●● ●●●	36	21

Trick IV — page 104

		Ben	You
Think of a number.	8	19	*
3rd line	8 8 + 100	138	

Trick V — page 104

		Ben	You
Think of a number.	8	7	*
4th line	2 8 + 26	40	

Trick VI — page 104

		Ben	You
Think of a number.	X	3	*
7th line	5X + 75	90	

Trick VII — page 104

		Al	Ed
Think of a number.	X	4	9
5th line	3X + 6	18	33

Two Extras

		P	Q
Think of a number.	X	1	7
	4X + 7	11	35

		R	S
Think of a number.	X	25	0
	3X + 150	225	150

* MANY POSSIBILITIES.

Tasks: Pupils determine the number thought of, using the results of a particular line of a Think-of-a-Number *game explored previously, plus two extra examples.*

Purpose: To teach pupils to use bag representations or the equivalent algebraic expressions.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Fred's idea is not really new—it was introduced on pupil page 102. The first example involves 1 bag and 6 counters; if this is 10 counters in all, then the bag must contain 4 counters. So the number thought of was 4. The third exercise has already been discussed in the Notes on pupil page 102.

Pupils may need to be reminded of the meaning of the various types of shorthand used in the last few exercises. Except for this notation the work is the same as before. If 5 bags and 75 counters make 90 counters then the 5 bags must contain 15 counters, for we must add 15 to 75 to get 90. Each bag contains 3 counters.

In the last trick, we can begin by imagining the total 225 counters. We set aside 150 of them and divide the remaining counters equally among the 3 bags. It is clear that 25 counters go into each bag.

For additional practice in handling algebraic expressions present several examples like the following:

If $\square = 5$ then $\square + 95 =$ ____

If $\square + 95 = 110$ then $\square =$ ____

Then try this form for keeping track of the results:

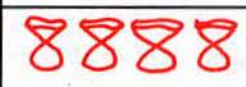
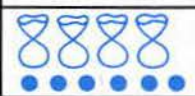
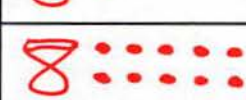
If $\square$ is	then $3\square + 25$ is
5	
6	
7	
8	
9	
10	

If $\square$ is	then $\frac{1}{2}\square + 10$ is
2	
4	
6	
8	
10	
5	
7	

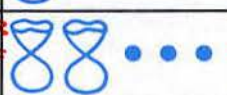
If	then $\square$ is
$5\square + 15 = 40$	
$7\square + 18 = 39$	
$\frac{1}{2}\square + 3 = 6$	
$\frac{1}{3}\square + \frac{2}{3} = 1$	
$\frac{1}{3}\square - \frac{1}{3} = 0$	

Here are incomplete records of some tricks Fred gave to the class. Please complete each record.

I.

Words	Pictures	Examples a	Examples b
Think of a number.		5	9
Multiply by 2.		10	18
Multiply by 2 again.		20	36
Add 6.		26	42
Subtract the number you thought of first.		21	33
Divide by 3.		7	11
Add 8.		15	19
Subtract the number you thought of first.		10	10

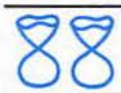
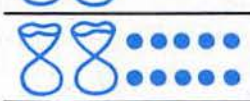
II.

Words	Pictures	Examples a	Examples b
THINK OF A NUMBER		11	6
ADD 3		14	9
ADD THE NUMBER YOU THOUGHT OF FIRST.		25	15
ADD 1		26	16
DIVIDE BY 2		13	8
SUBTRACT THE NUMBER YOU THOUGHT OF FIRST.		2	2

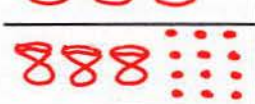
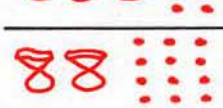
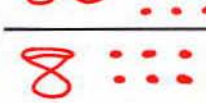
III.

Words	Shorthand	Examples a	Examples b
Think of a number.	x	10	4
Add 9.	x+9	19	13
Double.	2x + 18	38	26
Add the number you thought of first.	3x+18	48	30
Divide by 3.	x+6	16	10
Add 10.	x + 16	26	20
Double.	2x+32	52	40
Subtract the number you thought of first.	x+32	42	36
Subtract 25.	x + 7	17	11
Subtract the number you thought of first.	7	7	7

IV.

Pictures	Shorthand	Examples a	Examples b
	x	9	15
	2x	18	30
	2x+10	28	40
	x+5	14	20
	5	5	5

V.

Pictures	Shorthand	Examples a	Examples b
	x	4	7
	3x	12	21
	3x+12	24	33
	2x+12	20	26
	x+6	10	13
	6	6	6

Tasks: *Pupils reconstruct five* Think-of-a-Number *games by recording pictures, shorthand, and missing numbers.*

Purpose: To further establish the relationship between physical representations of the steps of *Think-of-a-Number* games and algebraic shorthand.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Exercise III requires students to begin by translating from words to x-shorthand. They may wish to first translate the words into pictures and then use x in place of a picture of a bag. Pupils may work back from the "26" in column *a* by reasoning: A bag and 16 counters makes 26 counters, so the bag holds 10 counters. Or if 16 added to x makes 26 then x is $26-16$. The number must have been 10. For the 26 in column *b*: if $2x+18=26$ then $2x=8$ and $x=4$.

In exercise II pupils must be able to work from pictures to words. In IV they must work from pictures to the x-shorthand. In V, they are given the numbers used by two players and must guess what the trick was.

To complete V, pupils have to know what two players do—one player's sequence would not be enough. For example, if they were told that a player thought of 4 and that the next number he wrote was 12, they would not know whether he had added 8 or multiplied by 3. But if one player goes from 4 to 12 and a second player goes from 7 to 21, *multiply by 3* fits both cases, but *add 8* does not. In the next step, going from 12 to 24 might be *double*, but this does not send 21 to 33. *Add 12* fits both players. The next step, *subtract the number first thought of*, is perhaps not easy to guess. Then comes *divide by 2* and finally *subtract the number you thought of first.*

To provide further practice, write out any but the last line of any example using shorthand and showing the total number of counters used at that point. Ask: What number had the player thought of in the beginning? Or write out the shorthand and give the number thought of. Ask: How many counters are used at this step?

Here is additional practice. Complete the following tables:

If x is	then $30x + 75$ is
10	
20	
30	
15	
25	
35	

If	then x is
$20+10x=50$	
$50+5x=50$	
$10+40x=50$	
$50x+0=50$	
$150x-100=50$	

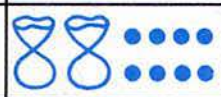
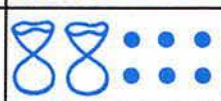
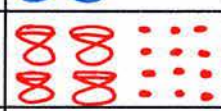
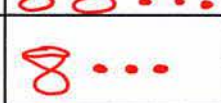
Reconstructions like the following should now offer a challenge:

If a is	then
0	$2a + 15 = \square$
1	$\square a + 10 = 20$
10	$14 + \square a = 19$
2	$20 + 5a = \square$
60	$30 - \frac{1}{2}a = \square$

If m is	then
8	$8m + 64 = \square$
4	$25m - \square = 0$
9	$\frac{1}{3}m + 9 = \square$
0	$12 - \frac{1}{5}m = \square$
10	$250 + 50m = \square$

WANTED: Endings for these tricks.

Trick A

Words	Pictures	Shorthand	Examples a.	Examples b.
Think of a number.		X	9	*
Add 4 to it.		X+4	13	
Double it.		2X+8	26	
Subtract 2.		2X+6	24	
MULTIPLY BY 2		4X+12	48	
DIVIDE BY 4		X+3	12	
SUBTRACT THE NUMBER YOU THOUGHT OF FIRST.		3	3	

*

Can you finish in two or three steps? In the step before the end, there should not be more than a single bag. When you say, "Subtract the number you thought of first," that gets rid of the bag.

Trick B

Words	Pictures	Shorthand	Examples a.	Examples b.
Think of a number.		X	*	*
Double it.		2X		
Add 9.		2X+9		
Add the number you thought of first.		3X+9		
*				

* MANY POSSIBILITIES.

Trick C

Words	Shorthand	Example
Think of a number.	X	*
Add 6.	X+6	
Double it.	2X+12	
*		

Trick D

Words	Shorthand	Example
Think of a number.	X	*
Triple it.	3X	
Add 8.	3X+8	
Subtract the number you thought of first.	2X+8	
*		

Tasks: *Pupils complete all four tricks, indicating pictures and shorthand at each step. They must find suitable endings for each trick.*

Purpose: To learn to make up endings to tricks. To provide further practice in relating the steps of tricks to words, pictures, and shorthand.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On pupil page 107 pupils are asked to make up their own endings to tricks. It has already been noted on pupil page 103 that to end a trick quickly requires considerable foresight and some experience. Therefore, it may not be wise to ask students to finish the trick in two or three steps. Rather, be pleased if they can finish it at all.

To reach a quick ending, students must learn to realize several things: first, that a trick is complete when no bag is left; second, that the step that accomplishes this is usually *take away the number first thought of;* third, that, for this last step to work, the previous step must reduce the number of bags to one; fourth, that it is usually possible to reduce the number of bags to one by dividing.

In the first example on pupil page 107 there are 2 bags and 2 counters at the end of the fourth step. A streamlined ending would be *divide by 2,* leaving 1 bag and 3 counters, and then *take away the number first thought of,* leaving no bags and 3 counters. The answer is 3, and the trick has been completed.

Another step is sometimes necessary in order to make division easier. For example, if a trick has reached a stage where there are 2 bags and 7 counters, it is not a good time to divide by 2, because this would result in 1 bag and 3½ counters. You can use the intermediate step *add 1* (or *subtract 1*) to keep the division simple, and then proceed with *divide by 2* and *subtract the number you thought of first.*

Here are some *What's My Rule?* games which involve algebraic expressions in the pattern indicators.

a	t
5	15
1	3
4	
	18
11	

$t = 3a$

c, d	y
3, 4	5
6, 3	
4, 4	
2,	13
, 15	8

$y = cd - 7$

a, e	f
3, 5	35
7, 2	
4, 4	
3,	20
, 2	38

$f = 50 - a$

u, v	w
5, 3	13
10, 11	
7,	20
, 7	25
, 12	36

$w = 2u + v$

a, b	y
20, 7	6
17, 4	
25, 8	
30,	16
21,	21

$y = a - 2b$

r, s	n
5, 2	11
6, 8	
10,	12
, 3	0
, 11	20

$n = 3r - 2$

Fred had another idea!

"If you show me part of a record of the kind of tricks and games we've been playing, I can often figure out the rest. You don't have to show me the words or pictures or shorthand.

"Here is a sample of the ways two tricks might get started. I'll explain my idea as we go along."

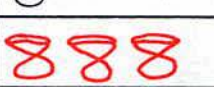
Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		x	5	3	1	11	7	0	25
ADD 5		x + 5	10	8	6	16	12	5	30

"Consider Anne's results. She thought of 5. By applying some rule not shown, she was led to 10. The rule might be to double the number or, perhaps, to add 5. Ben thought of 3. Doubling would lead to 6. But his result is 8. However, adding 5 would explain the results of both Anne and Ben.

"The rule of adding 5 would also explain the results recorded for Jane and Joe. Applying this rule to the numbers Matt and Betty thought of would lead to 6 and 30. And we can see that the number Ruth thought of would have been 7 because $7 + 5 = 12$.

"Now we can draw the pictures and show the idea in shorthand."

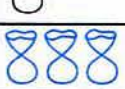
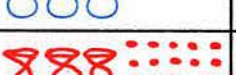
Here are the first steps of another trick:

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		x	0	100	7	11	10	12	25
TRIPLE IT		3x	0	300	21	33	30	36	75
				315				51	90

"From Anne's results in the first 2 rows, I know the rule could be add 0 or it could involve multiplication. Ruth thought of 10 and the rule led her to 30. It must be to multiply by 3 or, in other words, triple the number you thought of.

"Tripling would also explain Betty's results. Now I can draw pictures and write the shorthand for the second row. I can also find out the missing numbers in the first and second lines."

Next, consider the second and third rows.

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		x	0	100	7	11	10	12	25
Triple it.		3x	0	300	21	33	30	36	75
ADD 15		3x + 15	15	315	36	48	45	51	90

"Ben's results suggest right away that the rule might be to add 15. That rule explains the results given for Joe and Betty. Now I can complete the third line."

Notes on Pupil Page 108

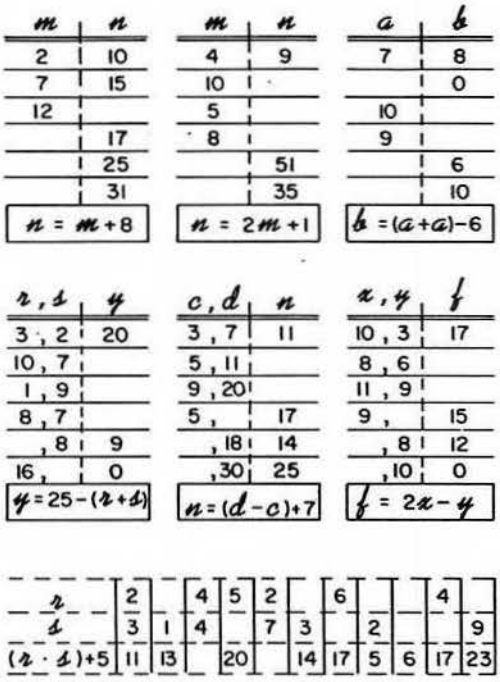
Tasks: Pupils reconstruct records of tricks.

Purpose: To teach pupils to reason from partial records so that they can reconstruct tables. To provide further practice in relating the steps of a development to words, pictures, and shorthand.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Perhaps this page might be studied and completed by individual pupils or pupil teams. This would be a good check on the extent of pupil understanding of the work so far and would give another hint about their ability to work independently.

Here are some suggestions for practice.



m	n
2	10
7	15
12	
	17
	25
	31

$n = m + 8$

m	n
4	9
10	
5	
8	
	51
	35

$n = 2m + 1$

a	b
7	8
	0
10	
9	
	6
	10

$b = (a + a) - 6$

x, y	y
3, 2	20
10, 7	
1, 9	
8, 7	
, 8	9
16,	0

$y = 25 - (x + y)$

c, d	n
3, 7	11
5, 11	
9, 20	
5,	17
, 18	14
, 30	25

$n = (d - c) + 7$

x, y	f
10, 3	17
8, 6	
11, 9	
9,	15
, 8	12
, 10	0

$f = 2x - y$

x	2		4	5	2		6			4	
y	3	1	4		7	3		2			9
$(x \cdot y) + 5$	11	13		20		14	17	5	6	17	23

Here is the record of a trick Fred says that he can reconstruct:

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
a. Think of a number.		x	11	7	3	8	1	6	9
b. DOUBLE IT.		2x	22	14	6	16	2	12	18
c. ADD 6.		2x+6	28	20	12	22	8	18	24
d. DIVIDE BY 2.		x+3	14	10	6	11	4	9	12
e. SUBTRACT THE NUMBER YOU THOUGHT OF FIRST.		3	3	3	3	3	3	3	3

"Consider lines a and b," Fred advises.

a. Think of a number.		x	11	7	3	8	1	6	9
b. DOUBLE IT.		2x	22	14	6	16	2	12	18

"In line a, I begin by drawing a bag. When anyone thinks of a number, I suppose that they put that many counters in the bag. Then I use Jerry's shorthand. I erase the top and bottom of the bag. The part that is left looks like the letter *x*.

"Next, I consider the results shown for Anne, Ben, and Matt. The rule is to double the number you thought of. So, I write this idea in words, pictures, and shorthand. Now, I can fill in the missing results for Ruth, Joe, and Betty."

			Anne	Ben	Matt	Jane	Ruth	Joe	Betty
b. Double it.		2x	22	14	6	16	2	12	18
c. ADD 6.		2x+6	28	20	12	22	8	18	24

"By studying the results for Ruth, Joe, and Betty, I know the rule must be to add 6."

c. Add 6.		2x+6	28	20	12	22	8	18	24
d. DIVIDE BY 2.		x+3	14	10	6	11	4	9	12

"By studying the results shown for Anne, Matt, and Ruth, I know the rule is to divide by 2."

a. Think of a number.		x	11	7	3		1	6	9
d. Divide by 2.		x+3	14	10	6	11	4	9	12
e. SUBTRACT THE NUMBER YOU THOUGHT OF FIRST.		3	3	3	3	3	3	3	3

"Consider the lines a, d, and e. The results for Ben, Joe, and Betty give the clue. The rule must be to subtract the number you thought of first. Now, you can complete the reconstruction of the entire record."

Tasks: Pupils reconstruct records of tricks.

Purpose: To teach pupils to reason from partial records so that they can reconstruct tables. To provide further practice in relating the steps of a development to words, pictures, and shorthand.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On this pupil page, Fred explains his reasoning for each step. Pupils may want to first see if they can complete the table at the top of the page before reading Fred's explanation.

In an ordinary crossword puzzle a word that may be very hard to guess at first may become obvious when other words around it have been filled in. The same is true of these number tricks. Look at Jane's work, for example. She has nothing at all in the first three rows. It is necessary to work back from Jane's fourth row (row *d*) by using information provided by the other students. Look at rows *b* and *c*:

						Joe	
b	22	14	6			12	
c					8	18	24

The only link between these rows is given by Joe's 12 and 18. This suggests *add 6*, but it could also be *multiply by 1½*. Obviously we need more information. If we study rows *a* and *b* for Anne, Ben, and Matt, we learn that row *b* is double row *a*. This means that rows *b* and *c* now look like this:

					Ruth	Joe	Betty
b	22	14	6		2	12	18
c					8	18	24

We can now conclude that *add 6* is the step from *b* to *c*.

Here is more practice with *What's My Rule?* games.

a, *r*	*t*
8, 5	6
17, 9	
21, 6	
, 7	16
, 9	28

$t = 2(a - r)$

d, *h*	*n*
8, 2	17
24, 3	
, 5	16
30,	19
, 4	11

$n = 21 - (d \div h)$

s, *w*	*y*
3, 7	26
5, 4	
, 9	22
8,	34
, 2	35

$y = (s \times w) + s + 2$

a, *b*	*m*
4, 3	14
6, 8	
4,	0
, 7	11
30,	18

$m = 5a - 2b$

Ben made up some challenging tricks. Can you reconstruct the records?

I

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		X	10	0	9	5	12	8	15
ADD 3.		X+3	13	3	12	8	15	11	18
DOUBLE IT.		2X+6	26	6	24	16	30	22	36
ADD 4.		2X+10	30	10	28	20	34	26	40
DIVIDE BY 2.		X+5	15	5	14	10	17	13	20
SUBTRACT THE NUMBER YOU THOUGHT OF FIRST.		5	5	5	5	5	5	5	5

II

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		X	4	0	10	5	25	100	19
TRIPLE IT.		3X	12	0	30	15	75	300	57
ADD 6.		3X+6	18	6	36	21	81	306	63
SUBTRACT THE NUMBER YOU THOUGHT OF FIRST.		2X+6	14	6	26	16	56	206	44
DIVIDE BY 2.		X+3	7	3	13	8	28	103	22
Subtract the number you thought of first.		3	3	3	3	3	3	3	3

III

Pictures	A	B	C	D
	3	7	11	8
	13	17	21	18
	7	11	15	12
	4	4	4	4

IV

Shorthand	E	F	G	H
X	6	0	3	7
2X	12	0	6	14
2X + 20	32	20	26	34
X + 20	26	20	23	27
X + 2	8	2	5	9
2	2	2	2	2

V

Pictures	I	J	K	L
	2	4	15	0
	7	9	20	5
	5	5	5	5
	1	1	1	1

VI

Shorthand	M	N	O	P
X	5	13	9	11
X + 4	9	17	13	15
3X + 12	27	51	39	45
2X + 12	22	38	30	34
X + 6	11	19	15	17
6	6	6	6	6

Notes on Pupil Page 110

Tasks: Pupils reconstruct records of tricks.

Purpose: To teach pupils to reason from partial records so that they can reconstruct tables. To provide further practice in relating the steps of a development to words, pictures, and shorthand.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: To anticipate pupils' thought processes you may wish to work through the exercises on scratch paper or in a pupil book. Exercise I is fairly straight-forward except for the fact that Betty's column cannot be completed until all the steps up to the last are known.

Exercise II is a bit more difficult, since the top row is empty. You might suspect that the second step is *triple it* from the fact that all the given numbers in the second row are multiples of 3; but you could not be certain. The third row, however, solves this problem; $3x+6$ is given, and each number is 6 more than the corresponding number in the second row. The shorthand for the second row, then, must be $3x$.

The fourth step presents no great difficulty and supplies enough information to begin filling in Jane's column. If 16 represents the total of 2 of Jane's bags and 6 counters, then each of her bags must contain 5.

Filling in the last row with 3's makes it possible to fill in the fifth row. If subtracting Anne's number, 4, from the number in the fifth row leaves 3, then the number in the fifth row must be 7. Finding the fifth-row entries in this way makes it possible to deduce the controlling instruction: *divide by 2.*

Filling in the last row with 3's also gives us clues to Joe's and Betty's numbers. If 103 minus Joe's number is 3, then Joe's number must be 100. If 22 minus Betty's number is 3, then her number must be 19.

If it seems likely that a large part of your class will have difficulty completing the tables, you may wish to put one or two on the chalkboard and let everyone contribute their hypotheses and insights. Unless someone is simply guessing at random, do not insist on an explanation before you try his suggestion. At this level, pupils intuit much more about such mathematical operations than they can possibly express in words or symbols. It is more important at this point for pupils to follow these intuitions to their logical conclusions than to justify them verbally in advance.

Fred repeats his claim:

"Some of these guessing games are becoming more complicated. But, if you give me the shorthand and the results in any row, I can still tell the number each player thought of first."

	Shorthand	a	b	c	d	e	f	g	h	i
A. One row of results.	$X + 8$	18	19	20	8	100	16	39	29	9
The number each player thought of first.	X	10	11	12	0	92	8	31	21	1
B. One row of results.	$2X + 5$	15	35	55	5	7	9	115	91	67
The number each player thought of first.	X	5	15	25	0	1	2	55	43	31
C. One row of results.	$4X - 3$	1	9	17	97	37	397	25	41	57
The number each player thought of first.	X	1	3	5	25	10	100	7	11	15
D. One row of results.	$10 + X$	12	20	11	60	10	21	39	35	17
The number each player thought of first.	X	2	10	1	50	0	11	29	25	7
E. One row of results.	$3X - 6$	15	0	30	24	9	144	36	69	84
The number each player thought of first.	X	7	2	12	10	5	50	14	25	30
F. One row of results.	$200 - X$	195	199	125	200	193	190	50	100	19
The number each player thought of first.	X	5	1	75	0	7	10	150	100	181
G. One row of results.	$25X + 25$	225	125	50	275	150	25	400	200	250
The number each player thought of first.	X	8	4	1	10	5	0	15	7	9
H. One row of results.	$9+3X+5X$	9	33	17	65	25	49	97	89	169
The number each player thought of first.	X	0	3	1	7	2	5	11	10	20
I. One row of results.	$4X+7-2X$	11	7	29	9	21	27	107	57	17
The number each player thought of first.	X	2	0	11	1	7	10	50	25	5
J. One row of results.	$9+2X+5X$	65	44	16	58	9	72	86	184	79
The number each player thought of first.	X	8	5	1	7	0	9	11	25	10

Notes on Pupil Page 111

Tasks: *Pupils determine the number thought of, using the results of any line of the* Think-of-a-Number *games explored thus far. They do the same for two new examples.*

Purpose: To teach pupils to use pictorial representations or algebraic expressions which convey the same idea.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Each of the problems on pupil page 111 can be tackled in several different ways, a few of which are indicated below. *In no case should formal rules of algebra be taught or used as the basis for an explanation.*

One way to attack a problem such as *a* in exercise A is to state it this way: I think of a number. I add 8 to it. I get 18. What number was I thinking of? Problem *f* in exercise B might be stated: 2 bags and 5 counters make a total of 9 counters. How many counters are in each bag? Pupils can answer such questions by using their common sense.

Problem *c* in exercise C could be written as:

$$(4 \times \_\_) - 3 = 17$$

If $4x - 3$ counters are 17 counters then there must have been 20 counters before the 3 counters were subtracted. If $4x$ counters are 20 counters then x counters must be 5 counters.

In exercise H it is probably easiest to argue that $9 + 3x + 5x$ indicates 9 counters, 3 bags, and 5 bags, which is the same as 9 counters and 8 bags. In exercise I, we add 7 counters to 4 bags and then remove 2 bags. This leaves us with 2 bags and 7 counters, or $2x + 7$.

Most of the questions on this pupil page can be tackled in one of these ways.

Anne explains an interesting and useful result.

"If I think of any number, I can always get the same results in two ways that seem very different.

"I'll keep track of these ways in shorthand.

I'm not going to use bags or 'x' because 'n' or any other mark would be just as good, as long as we agree it means the number thought of first."

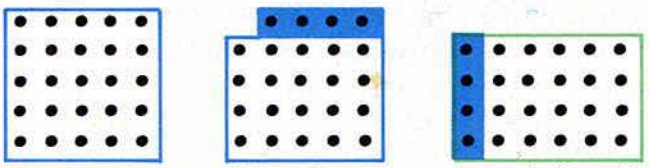
I. (Try some examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	A	B	C	D
Think of a number.	n	3	5	11	4	*1	*2	*6	*7
Multiply your number by itself.	n · n	9	25	121	16	1	4	36	49
Subtract 1 from the product.	(n · n) − 1	8	24	120	15	0	3	35	48
Add 1 to your number.	n + 1	4	6	12	5	*2	*3	*7	*8
Subtract 1 from your number.	n − 1	2	4	10	3	0	1	5	6
Multiply your results together.	(n + 1) · (n − 1)	8	24	120	15	0	3	35	48

* OTHER POSSIBLE ANSWERS

"For any number you can think of, is it true that $(n \cdot n) - 1 = (n + 1) \cdot (n - 1)$?"

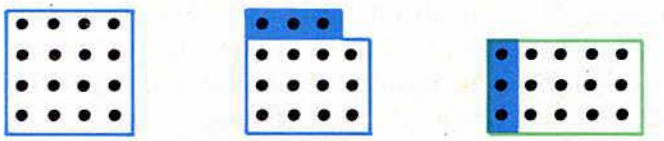
"You can make a picture to show why this happens. For example, the number Al thought of was 5."



5 · 5 (5 · 5) − 1 (5+1) · (5−1)

"After you take one dot away from 5 rows of 5 dots, you have just enough to make 4 rows of 6 dots.

"Jane's example works the same way."



4 · 4 (4 · 4) − 1 (4+1) · (4−1)

(n · n) − 1 = (n+1) · (n−1)

So does Mary's example.

11 · 11 (11 · 11) − 1 = (11+1) · (11−1)

"This is useful as a shortcut. If you want to know the answer to 21×19, you can think of it as $(20 + 1) \cdot (20 - 1)$. You also know that you can find the result by finding $(20 \cdot 20) - 1$. It is 399. Please use this same method in the following:

a. 31 x 29 = 899 d. 51 x 49 = 2,499

b. 13 x 11 = 143 e. 101 x 99 = 9,999

c. 41 x 39 = 1,599 f. 71 x 69 = 4,899

"It is also a shortcut when you want to find one less than 21×21. You know that $(21 \cdot 21) - 1 = (21 + 1) \times (21 - 1)$ or 22×20. The result is 440."

g. (31 · 31) − 1 = 960 i. (51 · 51) − 1 = 2,600

h. (41 · 41) − 1 = 1,680 j. (101 · 101) − 1 = 10,200

"So, of course, you can use this shortcut in the following:"

k. 31 · 31 = 961 m. 51 · 51 = 2,601

l. 21 · 21 = 441 n. 41 · 41 = 1,681

Notes on Pupil Page 112

Tasks: Pupils study the problem and complete the table (including their own examples). They further study pictures illustrating the problem and apply the shortcut methods they have learned to some examples in arithmetic.

Purpose: To help pupils explore a useful shortcut in multiplication through the study of arithmetic patterns. To help pupils understand how algebraic shorthand can represent the steps of such explorations. To introduce the use of "$\cdot$" as a sign for multiplication.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Pupil page 112 is based on the following "coincidence" in arithmetic: Think of any three consecutive numbers—say, 7, 8, and 9. The product of the first and third numbers ($7\times9=63$) is always 1 less than the square of the second number ($8\times8=64$). This holds for any three numbers—fractions and negative numbers included. (At this level, however, pupils are not ready to test it for any numbers except whole numbers.)

On this page we do four things.

1. We try various numbers to see if they confirm the above statement.
2. We draw pictures to show why it happens: If 8×8 is 1 more than 7×9, we ought to be able to remove 1 counter from an 8-by-8 array of counters and rearrange the rest into a 7-by-9 array.
3. We use the result as a shortcut in mental arithmetic.
4. We translate the idea into shorthand.

Several different elements of shorthand are involved.

On this page we replace the bag and abbreviated bag symbols by the letter n. We use it to indicate the middle number of a sequence of three consecutive numbers. Thus n would be 8 in the sequence 7, 8, 9. The students may think of 9 as *the number after 8*, but there is no symbol in algebra for "after." Instead of thinking of 9 as the number *after* 8, we think of it as *the number that is 1 greater than 8*. Hence we write $n+1$ for the number after n, and, following a similar line of reasoning, we write $n-1$ for the number before n.

Indicating multiplication poses a problem, since the familiar multiplication sign of arithmetic can be mistaken for the letter x. This can be solved in either of two ways: (1) we can use no sign at all and write $7n$ to indicate seven bags or seven times the number first thought of; or (2) we can replace the multiplication sign by a raised dot and write $7\cdot n$. The expressions $7n$ and $7\cdot n$ are both read "seven times n" or "seven n's." Similarly, we can read nn and $n\cdot n$ as "n times n" or "n n's."

We also introduce the use of parentheses to indicate the order in which operations are to be performed. Convention dictates that multiplications are to be performed before additions or subtractions. For example, $5\cdot5-1$ tells you to multiply 5 by 5 and then subtract 1 from this product.

$$\begin{aligned} 5\cdot5-1 &= 25-1 \\ &= 24 \end{aligned}$$

If the writer instead wants you to subtract before you multiply, he must use parentheses. Thus, $5\cdot(5-1)$ tells you to subtract 1 from 5 and then to multiply this difference by 5.

$$\begin{aligned} 5\cdot(5-1) &= 5\cdot4 \\ &= 20 \end{aligned}$$

Notice that in expressions like $(n\cdot n)-1$ and $(5\cdot5)-1$ the parentheses serve only to emphasize the normal order. In the expressions $(n+1)\cdot(n-1)$ and $(5+1)\cdot(5-1)$, however, they give priority to addition and subtraction.

Pupils may be amused by a mnemonic device: the signs ( ) are like blinders on a horse. Until you have worked out what is between them, you cannot see anything in the world outside them! You can hold your hands like parentheses at the side of your eyes to show that you see on the blackboard only what is between the parentheses.

For a fuller treatment of the order in which operations are to be performed, see page 45 of the introduction.

"This result is quite limited as a shortcut," Anne continued. "But it suggests further experiments that lead to similar shortcuts. Let's agree again that 'n' is shorthand for any number you think of. However, any letter you care to use is just as useful."

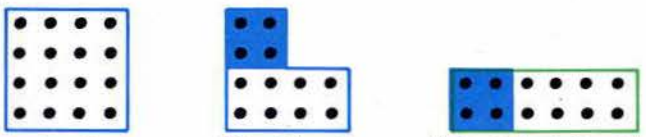
II.

Words	Shorthand	Ben	Al	Mary	Jane	(Examples of your own) E	F	G	H
Think of a number.	n	4	3	2	5	*6	*7	*8	*9
Multiply it by itself.	n · n	16	9	4	25	36	49	64	81
Subtract 4 from the product.	(n · n) − 4	12	5	0	21	32	45	60	77
Add 2 to your number.	n + 2	6	5	4	7	*8	*9	*10	*11
Subtract 2 from your number.	n − 2	2	1	0	3	4	5	6	7
Multiply your results together.	(n + 2) · (n − 2)	12	5	0	21	32	45	60	77

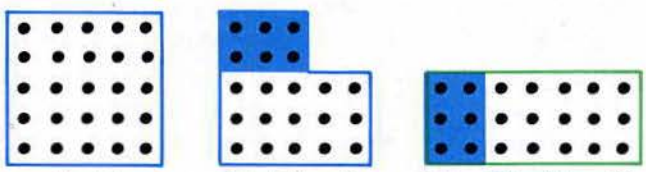
* OTHER POSSIBLE ANSWERS.

"For any number you can think of, is it true that $(n \cdot n) - 4 = (n + 2) \cdot (n - 2)$?"

"You can draw a picture to show why this happens. For example: Ben thought of 4.



"Jane thought of 5. You can draw a picture to show the results."



"You can use this result as a shortcut in the examples to the right."

a. $(22 \cdot 22) - 4 =$ 480

b. $(32 \cdot 32) - 4 =$ 1,020

c. $18 \cdot 22 =$ 396

d. $48 \cdot 52 =$ 2,496

III.

Words	Shorthand	Ben	Al	Mary	Jane	(Examples of your own) I	J	K	L
Think of a number.	n	7	4	3	5	*6	*8	*9	*10
Multilply it by itself.	n · n	49	16	9	25	36	64	81	100
Subtract 9 from that product.	(n · n) − 9	40	7	0	16	27	55	72	91
Add 3 to your number.	(n + 3)	10	7	6	8	*9	*11	*12	*13
Subtract 3 from your number.	(n − 3)	4	1	0	2	3	5	6	7
Multiply your results together.	(n + 3) · (n − 3)	40	7	0	16	27	55	72	91

* OTHER POSSIBLE ANSWERS

"For any number you can think of, is it true that $(n \cdot n) - 9 = (n + 3) \cdot (n - 3)$?"

Notes on Pupil Page 113

Tasks: Pupils study the experiments and complete the tables, including their own examples. They use the shortcut they find to solve arithmetic problems.

Purpose: To help pupils explore a useful shortcut in multiplication through the study of arithmetic patterns. To help pupils understand how algebraic shorthand can represent the steps in and conclusions of such explorations.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On pupil page 112 we squared a number ($n \cdot n$) and compared this result with the product of two other numbers: the number that is *one more* ($n+1$) and the number that is *one less* ($n-1$):

$$(n \cdot n) - (n+1) \cdot (n-1) = 1$$

On pupil page 113 we square a number and compare this result with the product of the number that is *two more* ($n+2$) and the number that is *two less* ($n-2$):

$$(n \cdot n) - (n+2) \cdot (n-2) = 4$$

As on pupil page 112, we draw pictures, we use the general statement as a shortcut for mental arithmetic, and we express the statement in shorthand. Any of the points mentioned on pupil page 112 may prove valuable here.

(Note that pupils at this level are not prepared to test the generalization for any numbers except the counting numbers greater than 1.) The next question is clearly: What about *three more* and *three less?* That, too, is studied on this pupil page. Your class may easily guess that on pupil page 114 they will be asked to consider the possibility of *four more* and *four less.* Perhaps they can carry on the investigation for themselves before reading pupil page 114. If so, excellent.

Here is an alternative development of the patterns we have been working with. Place the following table on the board:

$$(1+1) \times (1-1) = 1 \times 1 - \_\_$$
$$(2+1) \times (2-1) = 2 \times 2 - \_\_$$
$$(3+1) \times (3-1) = 3 \times 3 - \_\_$$
$$(4+1) \times (4-1) = 4 \times 4 - \_\_$$
$$(5+1) \times (5-1) = 5 \times 5 - \_\_$$

Surprising! Let's skip down the pattern to test its durability:

$$(10+1) \times (10-1) = 10 \times 10 - \_\_$$
$$(11+1) \times (11-1) = 11 \times 11 - \_\_$$
$$(100+1) \times (100-1) = 100 \times 100 - \_\_$$

The evidence is now quite convincing! Now, build another table using 2 in place of the recurring 1:

$$(2+2) \times (2-2) = 2 \times 2 - \_\_$$
$$(3+2) \times (3-2) = 3 \times 3 - \_\_$$
$$(4+2) \times (4-2) = 4 \times 4 - \_\_$$
$$(5+2) \times (5-2) = 5 \times 5 - \_\_$$

Jump ahead for random testing:

$$(10+2) \times (10-2) = 10 \times 10 - \_\_$$
$$(12+2) \times (12-2) = 12 \times 12 - \_\_$$
$$(25+2) \times (25-2) = 25 \times 25 - \_\_$$

Something certainly is going on. Try substituting 3 for the recurring 2:

$$(3+3) \times (3-3) = 3 \times 3 - \_\_$$
$$(4+3) \times (4-3) = 4 \times 4 - \_\_$$
$$(5+3) \times (5-3) = 5 \times 5 - \_\_$$
$$(6+3) \times (6-3) = 6 \times 6 - \_\_$$
$$(13+3) \times (13-3) = 13 \times 13 - \_\_$$

Anne gave another example:

(Examples of your own)

IV Words	Shorthand	Ben	Al	Mary	Jane	M	N	O	P
Think of any number.	n	5	10	4	8	*6	*7	*9	*11
Multiply it by itself.	n · n	25	100	16	64	36	49	81	121
Subtract 16 from that product.	(n · n) − 16	9	84	0	48	20	33	65	105
Add 4 to your number.	n + 4	9	14	8	12	*10	*11	*13	*15
Subtract 4 from your number.	n − 4	1	6	0	4	2	3	5	7
Multiply your results together.	(n + 4) · (n − 4)	9	84	0	48	20	33	65	105

*OTHER POSSIBLE ANSWERS.

"For any number you can think of, is it true that $(n \cdot n) - 16 = (n + 4) \cdot (n - 4)$?"

"Now, let's summarize our results," Anne suggested. "On this and the last two pages, we have found the following which we can test together."

	The number Ben thought of was 5	The number Mary thought of was 8
A. $(n \cdot n) - 1 = (n+1) \cdot (n-1)$	$25 - 1 = 6 \cdot 4$	$64 - 1 = 9 \cdot 7$
B. $(n \cdot n) - 4 = (n+2) \cdot (n-2)$	$25 - 4 = 7 \cdot 3$	$64 - 4 = 10 \cdot 6$
C. $(n \cdot n) - 9 = (n+3) \cdot (n-3)$	$25 - 9 = 8 \cdot 2$	$64 - 9 = 11 \cdot 5$
D. $(n \cdot n) - 16 = (n+4) \cdot (n-4)$	$25 - 16 = 9 \cdot 1$	$64 - 16 = 12 \cdot 4$

"Then the following group of statements ought to be true. Let's test them."

	If the number thought of is 9	If the number thought of is 4
E. $n \cdot n = (n+1) \cdot (n-1) + 1$	$81 = (10 \cdot 8) + 1$	$16 = (5 \cdot 3) + 1$
F. $n \cdot n = (n+2) \cdot (n-2) + 4$	$81 = (11 \cdot 7) + 4$	$16 = (6 \cdot 2) + 4$
G. $n \cdot n = (n+3) \cdot (n-3) + 9$	$81 = (12 \cdot 6) + 9$	$16 = (7 \cdot 1) + 9$
H. $n \cdot n = (n+4) \cdot (n-4) + 16$	$81 = (13 \cdot 5) + 16$	$16 = (8 \cdot 0) + 16$

"The following statements must also be true," Anne asserted.

	If the number thought of is 7	If the number thought of is 10
I. $(n+1) \cdot (n-1) = (n \cdot n) - 1$	$8 \cdot 6 = 49 - 1$	$11 \cdot 9 = 100 - 1$
J. $(n+2) \cdot (n-2) = (n \cdot n) - 4$	$9 \cdot 5 = 49 - 4$	$12 \cdot 8 = 100 - 4$
K. $(n+3) \cdot (n-3) = (n \cdot n) - 9$	$10 \cdot 4 = 49 - 9$	$13 \cdot 7 = 100 - 9$
L. $(n+4) \cdot (n-4) = (n \cdot n) - 16$	$11 \cdot 3 = 49 - 16$	$14 \cdot 6 = 100 - 16$

Notes on Pupil Page 114

Tasks: *Pupils study the experiments and complete the tables, including their own examples. They use the shortcut they find to solve arithmetic problems.*

Purpose: To help pupils explore a shortcut in multiplication through the study of arithmetic patterns. To help pupils understand how algebraic shorthand can help represent the steps and conclusions of such explorations.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On this pupil page, after an investigation of *four more* and *four less*, the students are presented with a summary of the four general patterns they have been following. These patterns are listed in rows A, B, C, and D together with numerical examples that demonstrate what the shorthand means. In rows E, F, G, and H the same facts are stated in a slightly different way. Instead of saying that 9×7 is 1 less than 8×8, we say that 8×8 is 1 more than 9×7. In rows I, J, K, and L we repeat the shorthand of A, B, C, and D, but we reverse the order. Instead of saying that $25-1=6\times4$, we say that $6\times4=25-1$.

In other words, the twelve rows A through L assemble the facts already known and state the same facts in three different forms. Obviously, we could go on like this for a long time, with *five more* and *five less*, then *six more* and *six less*, and so on. We would notice a pattern; we would find ourselves able to guess the next result.

Put these variations on the board to help your class discover the patterns:

$$(5+1)\times(5-1)=5\times5-\_\_\_$$
$$(5+2)\times(5-2)=5\times5-\_\_\_$$
$$(5+3)\times(5-3)=5\times5-\_\_\_$$
$$(5+4)\times(5-4)=5\times5-\_\_\_$$
$$(5+5)\times(5-5)=5\times5-\_\_\_$$

and:

$$(7+1)\times(7-1)=7\times7-\_\_\_$$
$$(7+2)\times(7-2)=7\times7-\_\_\_$$
$$(7+3)\times(7-3)=7\times7-\_\_\_$$
$$(7+4)\times(7-4)=7\times7-\_\_\_$$
$$(7+5)\times(7-5)=7\times7-\_\_\_$$

and so on—until everyone has got it.

Now put it to work and test it at the same time:

$$\begin{aligned}21\times19&=(20+1)\times(20-1)\\&=(20\times20)-1\\&=400-1\\&=399\end{aligned}$$

There are similar table extensions that require more study to find a rule. How is it possible to find a solution without actually going through the indicated manipulations?

$$(1\times2)-(0\times1)=2$$
$$(2\times3)-(1\times2)=4$$
$$(3\times4)-(2\times3)=6$$
$$(4\times5)-(3\times4)=8$$

One rule for this table might be: The difference must be twice the first number. Skip to test the rule:

$$(8\times9)-(7\times8)=\_\_\_$$

By the rule, the difference should be 8×2. Try another pattern:

$$(1\times3)-(0\times2)=3$$
$$(2\times4)-(1\times3)=5$$
$$(3\times5)-(2\times4)=7$$
$$(4\times6)-(3\times5)=9$$

The rule for this table might be: The difference is one more than twice the first number or one less than twice the fourth number. Test it:

$$(9\times11)-(8\times10)=\_\_\_$$

Finally, Anne ties these ideas together into a handy shortcut.

"There is a pattern running through the examples on the last three pages," Anne explained. "It is something like a game that might be called 'Think-of-Any-Two-Numbers.' Here's the way it works."

Think-of-Any Two Numbers	Shorthand	Ben	Ruth	Ellen	Roy	(Your own examples) A	B	C
The larger number.	b	7	6	10	15	*2	*3	*4
The smaller number.	s	3	4	7	10	1	1	1
The sum of your numbers.	b + s	10	10	17	25	3	4	5
The difference of your numbers.	b − s	4	2	3	5	1	2	3
The product of the sum and difference of your numbers.	(b+s)·(b−s)	40	20	51	125	3	8	15
The product of the larger number multiplied by the larger number.	b · b	49	36	100	225	*4	*9	*16
The product of the smaller number multiplied by the smaller number.	s · s	9	16	49	100	1	1	1
The difference of the products.	(b · b)−(s · s)	40	20	51	125	3	8	15

* OTHER POSSIBLE ANSWERS.

"For any pair of numbers you think of, $(b + s) \cdot (b - s) = (b \cdot b) - (s \cdot s)$."

"Many pairs of factors," Anne explained, "can be rewritten in the form of $(b + s) \cdot (b - s)$ and in the form of $(b \cdot b) - (s \cdot s)$. Often, these forms point to important shortcuts. Sometimes they do not help at all."

	The product of the pair of factors.	The product rewritten in the form of $(b + s) \cdot (b - s)$ or $(b - s) \cdot (b + s)$	The product rewritten in the form of $(b \cdot b) - (s \cdot s)$		
A.	12 · 8	= (10+2) · (10−2)	= (10 · 10) − (2 · 2)	= 100 − 4	= 96
B.	10 · 16	= (13−3) · (13+3)	= (13·13) − (3·3)	= 169 − 9	= 160
C.	21 · 19	= (20+1) · (20−1)	= (20·20) − (1·1)	= 400 − 1	= 399
D.	22 · 18	= (20+2) · (20−2)	= (20·20) − (2·2)	= 400 − 4	= 396
E.	35 · 25	= (30+5) · (30−5)	= (30·30) − (5·5)	= 900 − 25	= 875
F.	53 · 47	= (50+3) · (50−3)	= (50·50) − (3·3)	= 2,500 − 9	= 2,491
G.	56 · 44	= (50+6) · (50−6)	= (50·50) − (6·6)	= 2,500 − 36	= 2,464
H.	109 · 91	= (100+9) · (100−9)	= (100·100) − (9·9)	= 10,000 − 81	= 9,919
I.	79 · 81	= (80−1) · (80+1)	= (80·80) − (1·1)	= 6,400 − 1	= 6,399

Make up examples of your own.

Notes on Pupil Page 115

Tasks: *Pupils study the experiments and complete the tables, including their own examples. They use the shortcut they find to solve arithmetic problems.*

Purpose: To help pupils explore a shortcut in multiplication through the study of arithmetic patterns. To help pupils understand how algebraic shorthand can represent the steps of such explorations.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Rows I, J, K, and L of pupil page 114 begin like this:

$$(n+1)\cdot(n-1)=n\cdot n-1$$
$$(n+2)\cdot(n-2)=n\cdot n-4$$
$$(n+3)\cdot(n-3)=n\cdot n-9$$
$$(n+4)\cdot(n-4)=n\cdot n-16$$

Is there a pattern here? What should the next sentence in this sequence be? It is easy to guess that it will begin with:

$$(n+5)\cdot(n-5)=n\cdot n-\_\_\_$$

But what number should finish it? A pupil may say: Why, 1, 4, 9, and 16 are the square numbers: 1×1, 2×2; 3×3; 4×4. So the next number must be 5×5, which is 25. Another pupil may notice the steps in the growth of the sequence 1, 4, 9, 16:

Sequence	1	4	9	16
Differences	3	5	7	

The next difference should be 9, and the next number should be 25, which is quite correct! To be sure that this pupil also sees that $25=5\times5$, we might ask: Do you see anything about the factors of 1, 4, 9, and 16 that suggest a pattern? Or we might ask: Have we ever before met any sequence that grew by the odd numbers 3, 5, 7, 9 . . . ? The student may remember (or be helped to remember) the first exercise on pupil page 101. One way or another, the pupil ought to see the pattern:

$$(n+1)\cdot(n-1)=n\cdot n-1\cdot1$$
$$(n+2)\cdot(n-2)=n\cdot n-2\cdot2$$
$$(n+3)\cdot(n-3)=n\cdot n-3\cdot3$$
$$(n+4)\cdot(n-4)=n\cdot n-4\cdot4$$
$$(n+5)\cdot(n-5)=n\cdot n-5\cdot5$$

Once pupils see this pattern, they will see that:

n plus any number
times
n minus that number
equals
n times n
minus
that number times that number

It would be helpful if we had some shorthand for *any number*. Why not a? Then:

$n+a$
times
$n-a$
equals
n times n
minus
a times a

or, completely in shorthand:

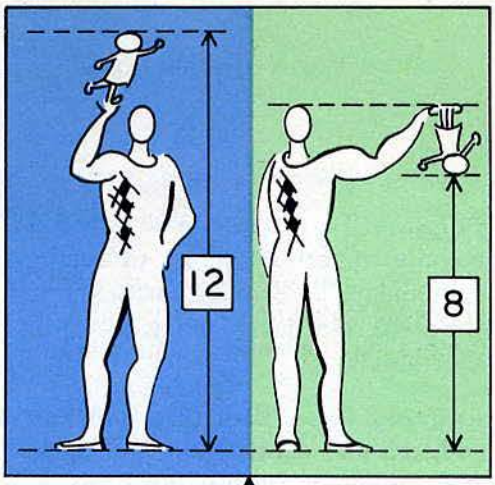
$$(n+a)\cdot(n-a)=n\cdot n-a\cdot a$$

On pupil page 115, we have not actually used n and a. We have used b for the bigger number and s for the smaller, so the result appears as

$$(b+s)\cdot(b-s)=b\cdot b-s\cdot s$$

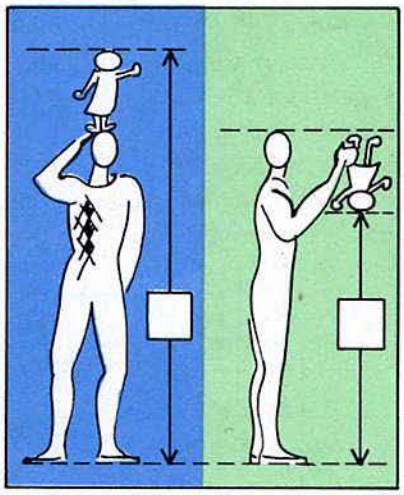
The meaning is exactly the same.

Guessing Games in the Land of Giant Acrobats and Dwarf Acrobats.



In each team of acrobats, giants are the same height and dwarfs are the same height. (All heights are given in numbers of feet.)

An example: The height of a giant and a dwarf together is 12 feet. A giant's height less that of a dwarf is 8 feet. How tall is each?

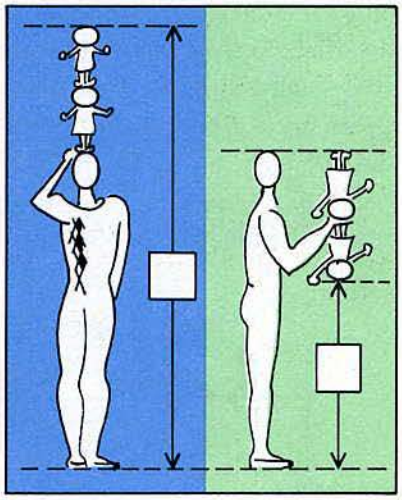


Ben and Anne imagined different results:

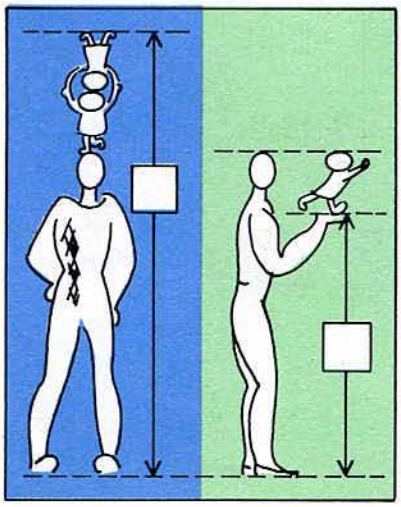
I.

The number of feet in the height of:	Shorthand	Example	Ben	Anne
a giant and a dwarf together	g + d	12	33	15
a giant less that of a dwarf	g − d	8	23	13
a giant	g	10	28	14
a dwarf	d	2	5	1

II. Fred, Dot, and Mary each imagined heights for members of another team of acrobats. Please guess the heights each person imagined.



The number of feet in the height of:	Shorthand	Fred	Dot	Mary
a giant and two dwarfs altogether	g + 2d	18	24	45
a giant less that of two dwarfs	g − 2d	6	16	25
a giant	g	12	20	35
a dwarf	d	3	2	5



III. Al, Jo, and Nan each imagined other heights for the arrangement of acrobats shown on the left. Please guess the heights each person imagined.

The number of feet in the height of:	Shorthand	Al	Jo	Nan
a giant and two dwarfs altogether	g + 2d	29	50	20
a giant less that of a dwarf	g − d	23	35	14
a giant	g	25	40	16
a dwarf	d	2	5	2

Notes on Pupil Page 116

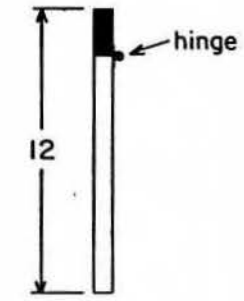
Tasks: *Pupils study the problem relating the heights of giants and dwarfs and then reconstruct the tables.*

Purpose: To provide a new context for relating algebraic shorthand to physical models. To provide further experience in reconstruction of a table.

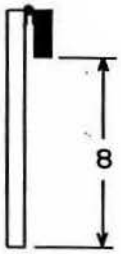
Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: The title of the game on pupil page 116 is important: A Guessing Game. The problems here resemble those found in a formal algebra course under the heading *simultaneous equations*. But the purpose of pupil pages 116-119 is *not* to teach pupils any formal methods of solving equations, but rather to get pupils used to expressions such as $g+d$ and $g-d$, so that they are able to use them without fear. We would like students to appreciate the meaning of $g+d$ and $g-d$, and the pictures of giants and dwarfs help to convey that meaning. Pupils learn to test their solutions. It is most important for any student of mathematics to check his solutions. Students who do not have this habit often talk nonsense.

Guessing can be aided by vision. The first problem can be illustrated by two objects hinged together:



Giant and Dwarf
g + d



Giant less Dwarf
g − d

It is surely evident that 2 dwarfs would fit into the space between heights 8 feet and 12 feet; and that the giant's height is midway between 8 feet and 12 feet. The above model can be improvised by folding a long strip of paper or by using pencils of unequal lengths to represent the giant and the dwarf.

In the same way, in Mary's problem there must be room for 4 dwarfs between the heights 25 feet and 45 feet. In Al's problem there is room for 3 dwarfs between the heights 23 feet and 29 feet.

Here is more practice with *What's My Rule?* games. Examples like the following provide good practice in understanding expressions like $g+d$ and $g-d$; $a-b$ and $a \cdot b$; etc.

	1	2	3	4	5	6	7	8
a−b			0	3	1	0	14	10
a	5	9						
b	3	2	4					
a · b	15			18	20	25	15	24

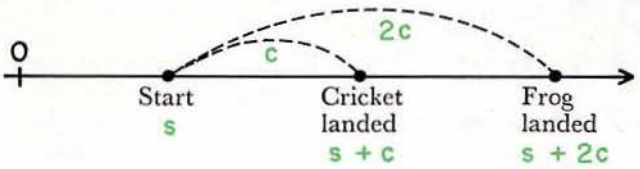
	9	10	11	12	13	14	15	16
r	6	7		7				
r+s		9			6	8		
r · s	24			21		16	9	0
s			3					
r−s			2		2		8	15

	17	18	19	20	21	22	23	24
x+y−1		7					9	
x	3		10	7				
x · y+5						14		
y	2	3			4			
x−y+1			10		2		7	8
x · y−2		13		19		7		28

Ruth's Guessing Game . . . (but she wasn't really guessing)

Ruth drew a number line on the board.

"A cricket and a frog started at the same point. The frog jumped twice as far as the cricket in the direction away from 0.



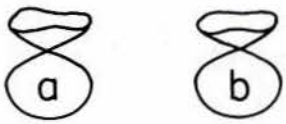
You must decide where each jumper started from and how far the cricket jumped. Tell me where each jumper landed. I'll tell you where you decided the jumpers started from and in what direction each jumped."

I.

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number that indicates the starting point.	s	5	10	10	0	14	1	7	8	50
Think of another number that indicates the size of the cricket's jump.	c	3	6	4	7	10	11	7	9	21
The frog jumps twice as far as the cricket.	2c	6	12	8	14	20	22	14	18	42
Tell where the cricket landed.	s + c	8	16	14	7	24	12	14	17	71
Tell where the frog landed.	s + 2c	11	22	18	14	34	23	21	26	92

Ruth told each player the numbers the player had thought of. Can you?

Ruth drew a pair of bags and labeled one "a" for Andy, and the other "b" for Bill. Then



she explained that Andy had more marbles in his bag than Bill had.

"Suppose you know how many each had. Use any two numbers you think of. Just tell me how many they have altogether and how many more Andy has than Bill."

II.

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number to indicate how many marbles Andy had.	a	10	9	1	8	9	20	17	35	83
Think of a number to indicate how many marbles Bill had.	b	3	6	0	2	8	8	0	29	49
How many marbles together?	a + b	13	15	1	10	17	28	17	64	132
How many more marbles does Andy have than Bill?	a − b	7	3	1	6	1	12	17	6	34

Again, Ruth told each player the numbers he had thought of. Can you?

(Make up your own story for the next example.)

III.

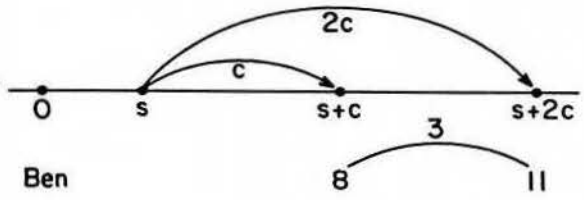
	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number.	m	3	2	5	1	9	1	10	11	15
Think of a second number.	n	5	2	1	7	0	1	15	7	15
Add the two numbers.	m + n	8	4	6	8	9	2	25	18	30
Add the second to 3 times the first.	3m + n	14	8	16	10	27	4	45	40	60

Tasks: Pupils study and reconstruct the problems.

Purpose: To teach pupils to reason from partial records to reconstruction of total patterns of development. To provide further practice in relating the steps of a development to words, pictures, and shorthand.

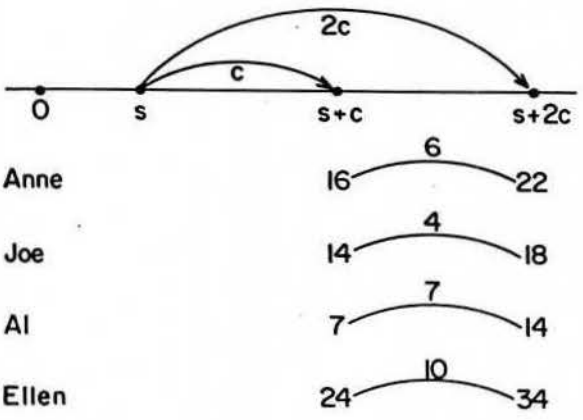
Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: The work on this pupil page closely resembles that on pupil page 116. The students should have little difficulty with the Frog and Cricket problem if they make a drawing to record the information.

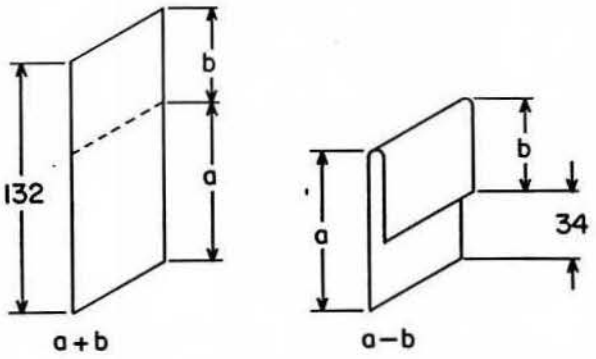


Ben tells us the Cricket lands at 8 and the Frog at 11. The distance between 8 and 11 is seen to be one cricket's jump.

Similarly, data for other problems can be recorded under the diagram to visualize the situation better. Thus:

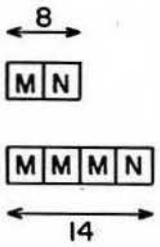


In working out most of the problems about Andy's bag and Bill's bag, pupils should be asked to guess the answers without any help at all. Where large numbers are involved, this may not be possible. In such cases, if the pupils have not already hit on some similar idea, point out to them that $a+b$ and $a-b$ can be pictured with a strip of paper in the same way that $g+d$ and $g-d$ were pictured. For example, Pete's numbers in exercise II can be illustrated:

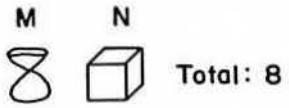


The picture suggests that there is room for $2b$ between 34 and 132. Since $132-34=98$ it follows that b must be 49. Bill had 49 marbles and Andy had 83 marbles.

Have pupils think of a way of picturing Ben's problem in exercise III. Perhaps they will think of blocks:



or bags and boxes:



A Gate Problem

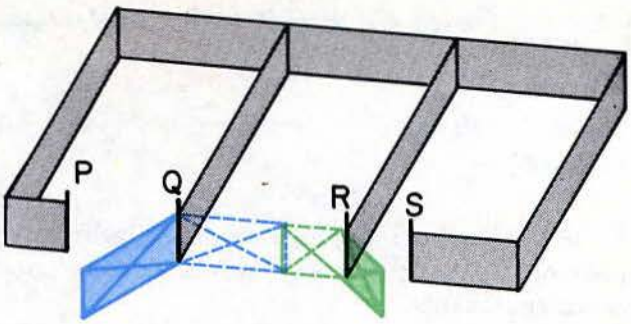
Ruth drew a diagram and explained:

"Each of you pretend you are a farmer with three fields. Each field is fenced in as shown. You have only two gates — one blue and one green. The gates may be of different lengths. You arrange them as shown.

"The blue gate alone can close off the field on the left. The green gate alone can close off the field on the right. The blue gate and green gate are both needed to close off the middle field. In this case, the gates meet but do not overlap.

"In order that we may talk about them, I have labeled four posts that are in a line — P, Q, R, and S. Since you know the length of each gate, you also know how far apart to place the posts.

"In the chart below, fill in *any two* of the questions marked A, B, C, D, E, F.



"Unless you trick me by answering a certain pair of questions, or by not giving me enough information to work on, I can figure out the length of each gate and the distances between the posts.

"Of course, if you answer the first two questions, my job is easy. Can you find a pair of questions that do not give me enough information?"

I.

Some of the players

What is the number of feet in:	Shorthand	Anne	Ben	Carl	Dan	Ellen	Fred	Gil	Harry	Ivan	June
A. the length of the blue gate	b	7	10	15	?	13	9	?	6	11	?
B. the length of the green gate	g	4	9	11	?	13	14	?	16	15	?
C. the distance from Q to R	b+g	11	19	26	27	26	23	19	22	26	?
D. the distance from P to R	2b+g	18	29	41	?	39	32	?	28	37	15
E. the distance from P to S	2b+2g	22	38	52	54	52	46	38	44	52	?
F. the distance from Q to S	b+2g	15	28	37	?	39	37	?	38	41	?

Did you discover the two players that stumped Ruth? They were the ones that answered questions C and E.

Compare all the results in row C with those in row E. Do you notice a pattern? Does it help explain why Dan and Gil stumped Ruth?

Did Harry and June give Ruth enough information to work on? HARRY DID - JUNE DIDN'T.

If some player answered only question C, would you know how he would have to answer question E? YES, TWICE ANSWER TO C.

If you were told that the blue gate is twice as long as the green one, can you complete a chart with only 1 question answered?

II.

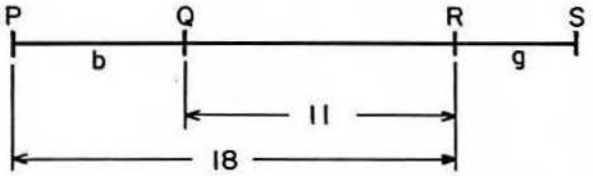
	Shorthand	K	L	M	N
A	b	18	10	12	22
B	g	9	5	6	11
C	b + g	27	15	18	33
D	2b + g	45	25	30	55
E	2b + 2g	54	30	36	66
F	b + 2g	36	20	24	44

Tasks: Pupils study the problems and then reconstruct the tables.

Purpose: To teach pupils to reason from partial records so that they can reconstruct missing entries in the table. To provide further practice in relating the steps of a development to words, pictures, and shorthand.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Anne's choice of numbers in exercise I can be visualized:



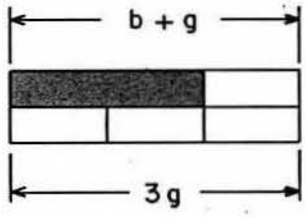
It is clear that b, from P to Q, is $(18-11)=7$. Most of the problems can be pictured this way.

Dan and Gil managed to stump Ruth by filling in only lines C and E. To see why, examine the information in these two lines. Line C gives us the combined length of one blue gate and one green gate. Line E tells us that if we had two blue gates and two green gates we could span the whole length PS—which is obvious from the drawing. In other words, all that line E does is double the number in line C; it tells us nothing new about either gate.

The drawing doesn't give us much help with Harry's problem, and students may not see any simple way to reason it out. Trial and error is probably the quickest way to arrive at an answer. We can find many ways of making $2b+g=38$. From all of these, we need one that makes $b+2g=38$. The work might go like this:

Guesses	Make $2b+g=28$	Find $b+2g$	Result
(1) $b=10$	$g=8$	26	Too small
(2) $b=11$	$g=6$	23	Even worse! Go in opposite direction
(3) $b=9$	$g=10$	29	Better than the first two guesses
(4) $b=8$	$g=12$	32	Still better
(5) $b=7$	$g=14$	35	Very close now
(6) $b=6$	$g=16$	38	This is it

In exercise II we are told that a blue gate is twice as long as a green gate. So a blue gate and a green gate have the same total length as three green gates.



In shorthand, $b+g=3g$. In column K we are told that $b+g=27$. So, $3g=27$. Therefore, $g=9$. Since the blue gate is twice as long as the green gate, $b=18$. Now we know the sizes of both gates, and the rest is easy.

Column L is also easy. We are told that the length of the green gate is 5 feet. The blue gate is twice as long, so it must be 10 feet.

In column M, we are told the combined lengths of one blue and two green gates. This is the same length as four green gates. (Pupils should draw a picture if they doubt this.) So, the green gate must be 6 feet long. Now, the column can be completed in the same way as column L.

Finally, in column N, since two blue and two green gates have the same length as six green gates, the green gate must be 11 feet long.

"Here's another game," Ruth began.

"Each of you think of any two numbers. Then tell me what you get for any two of the six expressions given below.

I will tell you what you would get for the other expressions. I will also tell you the two numbers you thought of."

I.

Shorthand

Player	one number x	other number y	x + y	2x + y	x + 2y	3x + y	3x + 2y	3x + 4y
Anne	3	5	8	11	13	14	19	29
Betty	7	2	9	16	11	23	25	29
Carl	5	5	10	15	15	20	25	35
Dan	10	9	19	29	28	39	48	66
Ellen	15	1	16	31	17	46	47	49
Fred	4	11	15	19	26	23	34	56
Gil	2	4	6	8	10	10	14	22

"Think of any two numbers (one larger than the other). The rest of the rules are the same. I will change the expressions. Please tell me what results you get for any two expressions.

II.

Shorthand

Player	bigger number b	smaller number s	b + s	b − s	b + 2s	2b − s	2b + s	3b − s
Anne	10	5	15	5	20	15	25	25
Betty	9	8	17	1	25	10	26	19
Carl	4	1	5	3	6	7	9	11
Dan	12	2	14	10	16	22	26	34
Ellen	15	5	20	10	25	25	35	40
Fred	6	0	6	6	6	12	12	18
Gil	7	4	11	3	15	10	18	17

"Again think of any two numbers (one larger than the other). Tell me what you get as results for two of the five statements. I will tell you the numbers you thought of."

III.

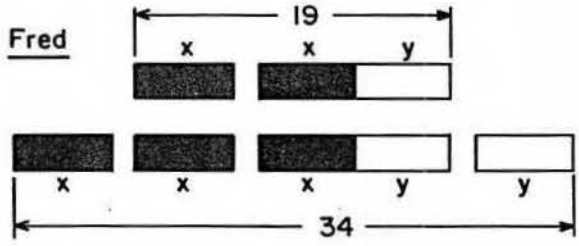
Words	Shorthand	Anne	Betty	Carl	Dan	Ellen	Fred	Gil
Think of a number.	p	6	7	10	12	20	11	7
Think of a smaller number.	q	2	5	3	7	9	5	0
Add your numbers.	p + q	8	12	13	19	29	16	7
Add the smaller number to twice the larger number.	2p + q	14	19	23	31	49	27	14
Subtract the smaller number from the larger number.	p − q	4	2	7	5	11	6	7
From five times the larger number subtract the smaller number.	5p − q	28	30	47	53	91	50	35
Add three times the larger number to twice the smaller number.	2q + 3p	22	31	36	50	78	23	21

Tasks: Pupils complete the three tables by studying the clues given.

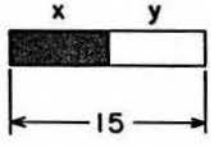
Purpose: To help pupils interpret algebraic expressions by using physical models. To help pupils reconstruct tables.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Exercise I is easy once pictures are drawn. Pupils will invent many different ways of tackling these problems. Here is one way Fred's problem might be approached:

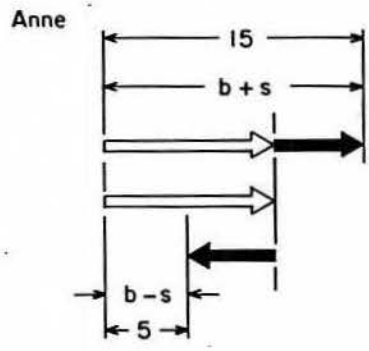


We see that the length increased from 19 to 34 when we added one x and one y block. Since $34-19=15$, it must be that $x+y=15$.



Compare this with the picture illustrating 19. An additional x makes the difference between 15 and 19. So, $x=4$. Then y must be 11.

The problems in exercise II can be pictured, if the students like, by giants (b for big) and dwarfs (s for small). Illustrations of addition and subtraction like those in Notes on Pupil Page 116 can be used. A more compact illustration is shown here. We represent b by a white arrow and s by a black arrow. When s is subtracted, this arrow points to the left.



We see that two black arrows fit into the space between 5 and 15. So, s must be 5. This means that $b=10$.

The problems in exercise III can be illustrated in the same ways as those in exercise II.

Trial and error is the easiest approach to Gil's problem; $5p-q=35$ suggests that $5p$ is greater than 35; if so, p must be greater than 7. Trying 8, we get $p=8$, $q=5$. Trying 9, we get $p=9$, $q=10$. Trying 10, we get $p=10$, $q=15$. Do any of these pairs of numbers make $3p+2q=21$?

$5p-q$	35	35	35
p	8	9	10
q	5	10	15
$3p+2q$	34	47	60

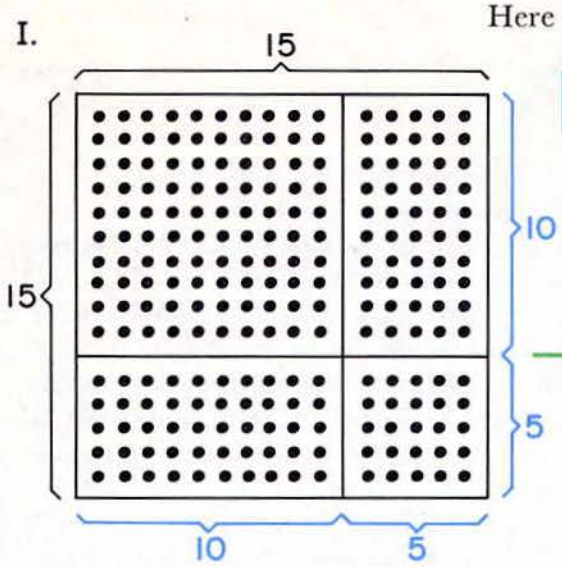
We see that we are getting farther and farther from 21. We try the opposite direction:

$5p-q$	(35)	35	35	35
p	7	8	9	10
q	0	5	10	15
$3p+2q$	(21)	34	47	60

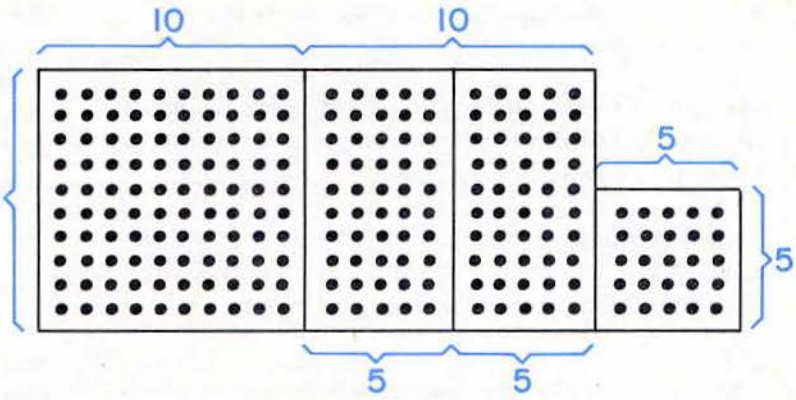
We have found the pair: $p=7$, $q=0$.

Here are hints to a useful shortcut.

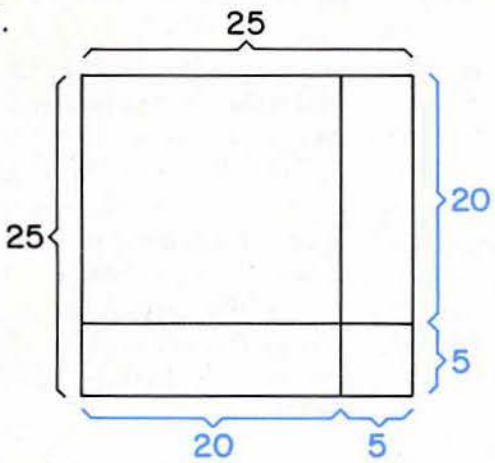
I.

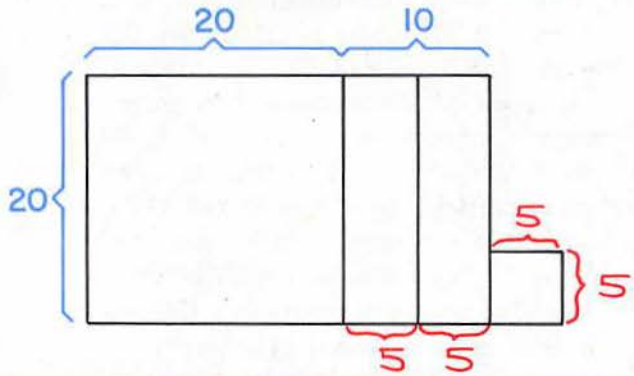


$15 \times 15 = 10 \cdot (10 + 10) + 25 = (10 \cdot 20) + 25 = 225$



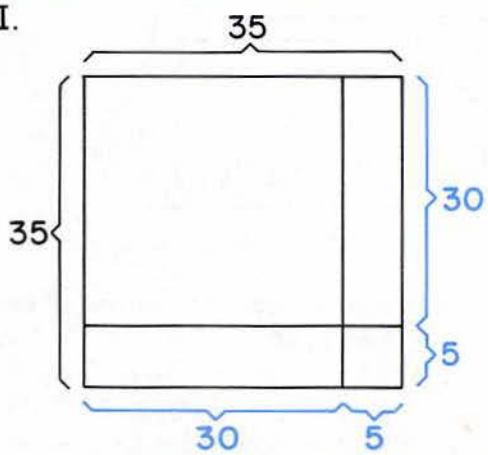
II.



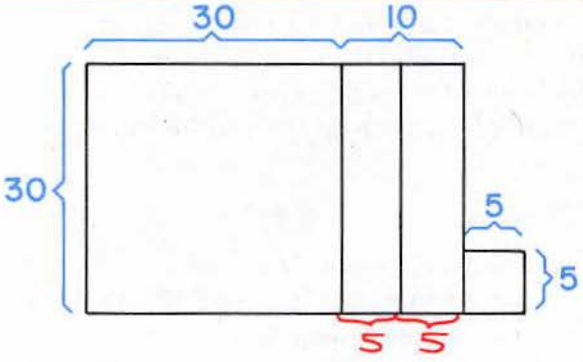


$25 \cdot 25 = 20 \cdot (20 + 10) + 25 = (20 \cdot 30) + 25 = 625$

III.



$35 \times 35 = 30 \cdot (30 + 10) + 25 = (30 \cdot 40) + 25 = 1{,}225$



IV.

a. $15 \cdot 15 = 10 \cdot (10 + 10) + 25 = (10 \cdot 20) + 25 = 225$

b. $25 \cdot 25 = 20 \cdot (20 + 10) + 25 = (20 \cdot 30) + 25 = 625$

c. $35 \cdot 35 = 30 \cdot (30 + 10) + 25 = (30 \cdot 40) + 25 = 1{,}225$

d. $45 \cdot 45 = 40 \cdot (40 + 10) + 25 = (40 \cdot 50) + 25 = 2{,}025$

e. $55 \cdot 55 = 50 \cdot (50 + 10) + 25 = (50 \cdot 60) + 25 = 3{,}025$

Can you guess the results for:

$65 \times 65 = 4{,}225$

$85 \times 85 = 7{,}225$

Test your guesses.

Tasks: *Pupils complete three mathematical sentences represented by diagrams. They then complete similar mathematical sentences, imagining the pictures or drawing them if needed.*

Purpose: To help pupils discover a shortcut for multiplying a number by itself if its final digit is 5. To relate such a multiplication to diagrams.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations; geometry.

The Lesson: There is a well-known rule for multiplying a number by itself if its final digit is 5. Suppose, for example, we wish to find 75×75. The rule works like this: 75 lies between 70 and 80. Multiply these two numbers: $70 \times 80 = 5600$. To this product add 25. This gives 5625, the correct answer.

On pupil page 120 we show why this rule works. We start with the simplest example, 15×15. First we draw 15 rows of 15 dots to represent 15×15. Then we cut this up, and put the pieces together so that we have 10 rows of 20 dots, *plus* a square containing 25 dots. This shows that 15×15 is the result of adding 25 to 10×20.

We then consider 25×25. We do not draw all the dots, but they should be imagined. We show how to cut the picture up and put it together again so that it shows 20 rows of 30 dots, together with a square of 25 dots. This work illustrates the result, $25 \times 25 = (20 \times 30) + 25$.

The third picture illustrates $35 \times 35 = (30 \times 40) + 25$.

The wide usefulness of this rule is explored on the next page.

Consider giving further work with shorthand and tables such as the following:

Example A

r	3	4					
s−r	2				1	5	7
s+r	8		10		7	11	7
s	5	5	8				

Example B

a	3	4	8	6			
b	7	1			8		
a+b			10			5	1
2a+2b				16	22	10	2

Example C

r	2	3	2	3				
r·c		9				16	10	
c	5			6	4			19
c−r			5		1	0	9	18

Example D

x·y−x					0	12	20
x	5	7	4	3			
y	3	2	1		7		
x·(y−1)				9		12	20

Example E

r	8					26		
s	4				2			
r·s		48	15		44		29	
r−s			14	6				16
r÷s				4		13		
r+s		16		25	18		30	18

You can put the correct answers down as fast as you can write:

a.	b.	c.	d.	e.	f.	g.	h.	i.
15 x 15 = 225	25 x 25 = 625	35 x 35 = 1,225	45 x 45 = 2,025	55 x 55 = 3,025	65 x 65 = 4,225	75 x 75 = 5,625	85 x 85 = 7,225	95 x 95 = 9,025

Now put this useful shortcut to work by combining it with $(b+s)\cdot(b-s) = (b\cdot b)-(s\cdot s)$

1. $23 \cdot 17 = (20+3)\cdot(20-3) = (20\cdot 20)-(3\cdot 3) = 400-9 =$ 391
2. $27 \cdot 23 = (25+2)\cdot(25-2) = (25\cdot 25)-(2\cdot 2) = 625-4 =$ 621
3. $39 \cdot 31 = (35+4)\cdot(35-4) = (35\cdot 35)-(4\cdot 4) = 1{,}225-16 =$ 1,209
4. $41 \cdot 39 = (40+1)\cdot(40-1) = (40\cdot 40)-(1\cdot 1) = 1{,}600-1 =$ 1,599
5. $26 \cdot 24 = (25+1)\cdot(25-1) = (25\cdot 25)-(1\cdot 1) = 625-1 =$ 624
6. $47 \cdot 43 = (45+2)\cdot(45-2) = (45\cdot 45)-(2\cdot 2) = 2{,}025-4 =$ 2,021
7. $98 \cdot 92 = (95+3)\cdot(95-3) = (95\cdot 95)-(3\cdot 3) = 9{,}025-9 =$ 9,016
8. $109 \cdot 101 = (105+4)\cdot(105-4) = (105\cdot 105)-(4\cdot 4) = 11{,}025-16 =$ 11,009
9. $82 \cdot 78 = (80+2)\cdot(80-2) = (80\cdot 80)-(2\cdot 2) = 6{,}400-4 =$ 6,396
10. $61 \cdot 49 = (55+6)\cdot(55-6) = (55\cdot 55)-(6\cdot 6) = 3{,}025-36 =$ 2,989

Now put this useful shortcut to work by combining it with $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$

11. $25 \cdot 26 = 25\cdot(25+1) = (25\cdot 25)+(25\cdot 1) = 625+25 =$ 650
12. $35 \cdot 38 = 35\cdot(35+3) = (35\cdot 35)+(35\cdot 3) = 1{,}225+105 =$ 1,330
13. $25 \cdot 24 = 25\cdot(25-1) = (25\cdot 25)-(25\cdot 1) = 625-25 =$ 600
14. $45 \cdot 43 = 45\cdot(45-2) = (45\cdot 45)-(45\cdot 2) = 2{,}025-90 =$ 1,935
15. $39 \cdot 35 = 35\cdot(35+4) = (35\cdot 35)+(35\cdot 4) = 1{,}225+140 =$ 1,365

Now take most of the steps without writing them down — if you can.

A.	B.	C.	D.	E.	F.	G.
34 x 35 = 1225 − 35 = 1,190	33 x 37 = 1225 − 4 = 1,221	53 x 47 = 2500 − 9 = 2,491	47 x 45 = 2025 + 90 = 2,115	65 x 66 = 4225 + 65 = 4,290	87 x 93 = 8100 − 9 = 8,091	71 x 59 = 4225 − 36 = 4,189

I $35 \cdot 35 = 1{,}225$	III $35 \cdot 36 = 1{,}260$	V $34 \cdot 36 = 1{,}224$	VII $31 \cdot 39 = 1{,}209$
II $45 \cdot 45 = 2{,}025$	IV $46 \cdot 45 = 2{,}070$	VI $48 \cdot 42 = 2{,}016$	VIII $46 \cdot 44 = 2{,}024$

Tasks: Pupils complete examples according to indicated patterns.

Purpose: To give pupils practice in using shortcuts to long multiplication problems, using patterns learned on earlier pupil pages.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: This page begins with a series of examples that apply the shortcut developed on pupil page 120. Exercises 1-10 require pupils to use the results of the investigation on pupil pages 112-115. You may wish to refresh their memory by asking questions such as: Is 6×4 more or less than 5×5? How much less? What is the difference between 14×12 and 13×13? Between 218×216 and 217×217? This last question is mental arithmetic. Students do not need to know what 217×217 is. They need only remember that the product of the end numbers, 218×216, is 1 less than the product of the middle number and itself, 217×217. In shorthand,

$$(n+1)\cdot(n-1)=n\cdot n-1$$

See if your pupils remember what happens with $n+2$ and $n-2$. What is the difference between 7×3 and 5×5? Between 22×18 and 20×20? This review should remind them that $(n+2)\cdot(n-2)$ is always the same as $n\cdot n-4$. Guide pupils along the line of thought followed on pupil pages 112-115, so that they again think through to the conclusion:

$$(n+a)\cdot(n-a)=n\cdot n-a\cdot a$$
$$(b+s)\cdot(b-s)=b\cdot b-s\cdot s$$

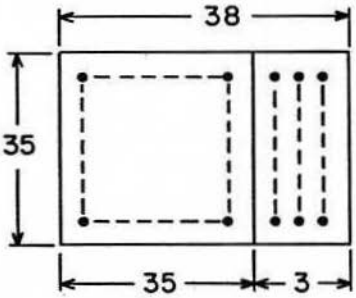
This should prepare them for these exercises.

In exercise 1 we wish to multiply 23 and 17. We first observe that 20 comes midway between these numbers. Since $23=20+3$ and $17=20-3$, we can predict that the product of 23 and 17 will be 9 less than $20\cdot20$. It is easier to find $400-9$ than it is to multiply 23 and 17 by the usual method.

In exercise 2 we wish to find $27\cdot23$. Midway between these numbers is 25. Since $27=25+2$ and $23=25-2$, we know that $27\cdot23$ will be 4 less than $25\cdot25$. We have already found $25\cdot25$ at the top of the page. It is 625. So $27\cdot23$ must be $625-4=621$.

All the other exercises in this section are solved in a similar way. For the b in the formula, we choose the number midway between the numbers we want to multiply. For example, in exercise 10 we must find $61\cdot49$. Midway between 49 and 61 is 55. Since $49=55-6$ and $61=55+6$, we know that $61\cdot49$ will be 36 less than $55\cdot55$.

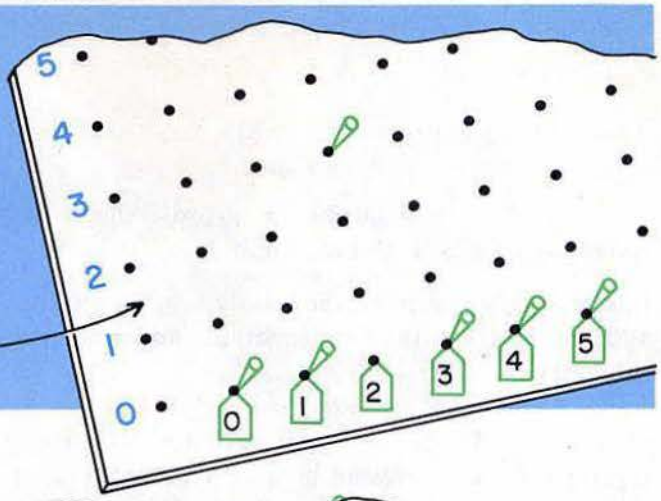
The next five exercises combine the result at the top of this pupil page with the distributive principle. For instance, when we want $35\cdot38$, we see that this is not far from $35\cdot35$. In fact, 35 rows of 38 dots can be broken into 35 rows of 35 dots and 35 rows of 3 dots. The results at the top of this pupil page tell us that $35\cdot35=1225$. So there must be 1225 dots in the square shown below and 105 dots in the tall rectangle at the right. The total number of dots in $35\cdot38$ must be the sum of these numbers: $1225+105$. All five exercises are similar.



For the mental arithmetic at the bottom of the pupil page, the rule of pupil page 120 is used in exercises I and II. The formula $(b+s)\cdot(b-s)=b\cdot b-s\cdot s$ is used in exercises B, C, F, G, and V-VIII. The distributive principle is used for in exercises A, D, E, III, and IV.

Using peg board, green pegs, and tags, Bill worked out a way of keeping track of the numbers people call out and the number he answers after applying his rule.

If someone calls out "2" and Bill answers "3," he finds the peg near the tag with 2 written on it. He moves the peg up to the row of holes opposite 3 on the left side of the board. It would look like the picture to the right.



"I'm thinking of a rule," Bill began.

"3," Ben called out.

"5," Bill answered. He recorded the answer. ⟶

"0," Helen called out.

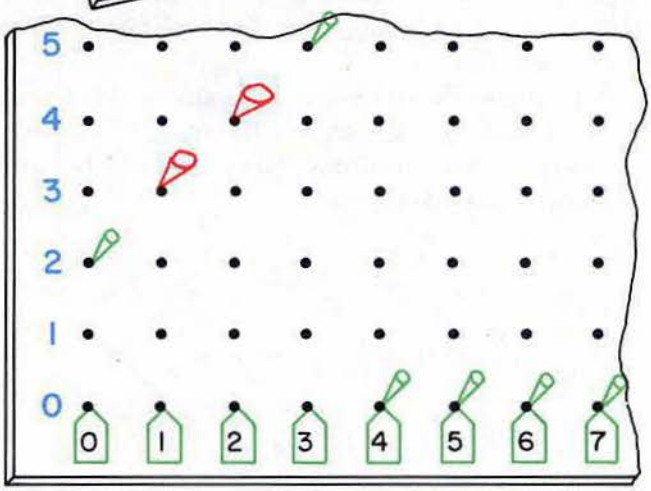
"2," Bill answered, and recorded the answer. ⟶

"1," Anne called out, "and I think you are going to answer 3." ⟶

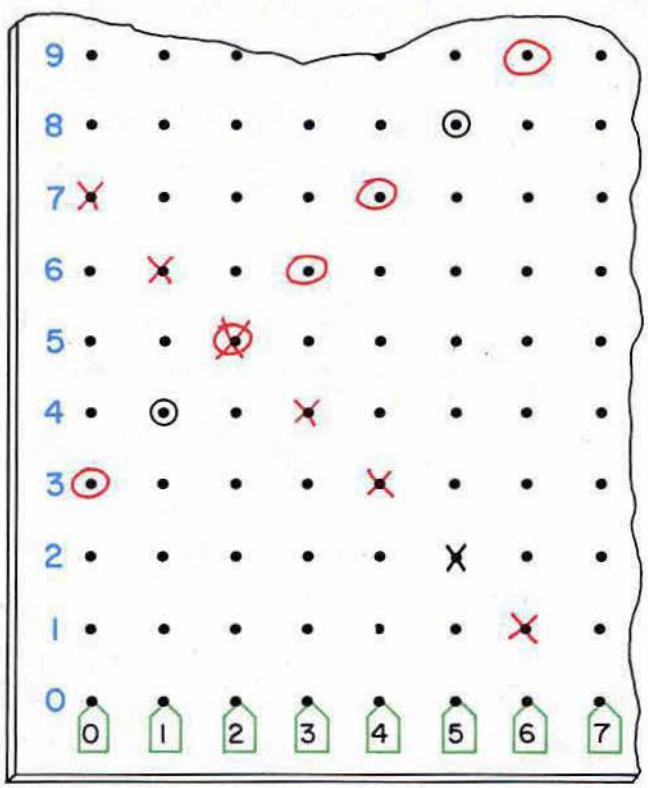
"That's right. You know my rule."

"If I say 2, will you say 4?" Al asked.

"Right. You know the rule also."



The first two results are recorded above. Please record the others.



On the left, keep a record of the games shown below. You will need to discover the secret rule for each game in order to complete it.

For the first game, instead of drawing pegs, circle the places where the pegs would go. The first two answers are already indicated for the first game. The first answer for the second game is indicated by an 'X'. Place 'X' marks where the pegs would be placed for the second game.

1.

Numbers called out	1	5	4	0	6	3	2
Bill's answers	4	8	7	3	9	6	5

2.

Numbers called out	5	3	6	0	4	1	2
Bill's answers	2	4	1	7	3	6	5

Notes on Pupil Page 122

Tasks: Pupils record four What's My Rule? *patterns on pegboard diagrams according to the instructions given.*

Purpose: To show pupils how a *What's My Rule?* pattern can be recorded on a grid.

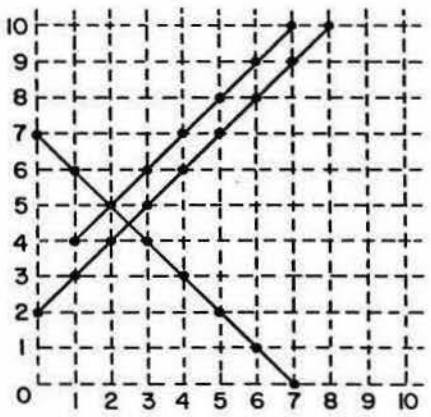
Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations; geometry.

The Lesson: In the *What's My Rule?* game on pupil page 122 when Anne called out 1, Bill's answer was 3. We show that Bill indicated his answer by placing a peg above the tag 1 opposite the number 3, as shown (in red).

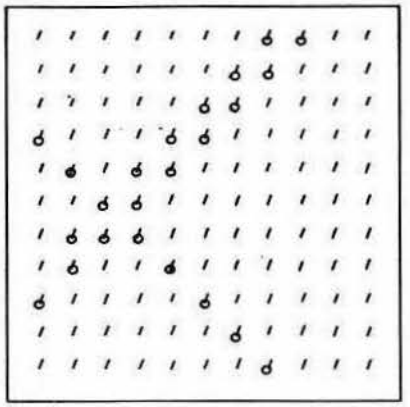
When Al called out 2 he thought Bill would answer 4—and he was right. We record this by beginning at the tag 2 and going up until we are even with 4. A peg (shown in red) is inserted.

In the tables at the bottom of this pupil page, the rule for the first game is *add 3*; for the second game it is *subtract from 7*.

Graph paper records of the *What's My Rule?* patterns look like this:



On the Geo-board, the record looks like this:



Have pupils or pupil teams invent other rules and represent the pattern on a peg board, graph paper, or Geo-board.

Here are additional *What's My Rule?* games for pupils to complete and diagram:

7	10	5	4		8
3	6			2	

4	2	0	5	3	7
8	6	4			

5	1	6	3	8	4
5	9	4			

8	6	4	5	7	9
4	2	0			

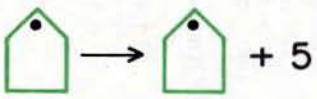
So far, all rules have led to records composed of points in a straight line. This will be true as long as simple addition, subtraction, multiplication, and division are used in the rule. However, when multiplication of a number by itself becomes involved, a drastic change occurs. For your own introduction to this idea, try graphing this game:

0	1	2	3	4	etc.
0	1	4	9	16	

The record is composed of points that seem to lie on a curve. No examples given in the development in Level D involve multiplying a number by itself. However, we are certainly leading in that direction. Some pupils may think up such rules that lead to this surprising development. So be forewarned!

Bill explains a shorthand way to write down rules after they are discovered.

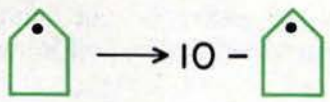
"When someone calls out a number, I imagine it as written on a tag at the bottom of the scoreboard. If my rule is to answer by saying a number that is five more than the number called out, I think this way:



"So, the green tag holds a place for any number you think of. I read that shorthand as:

'Tag goes to tag plus 5.'

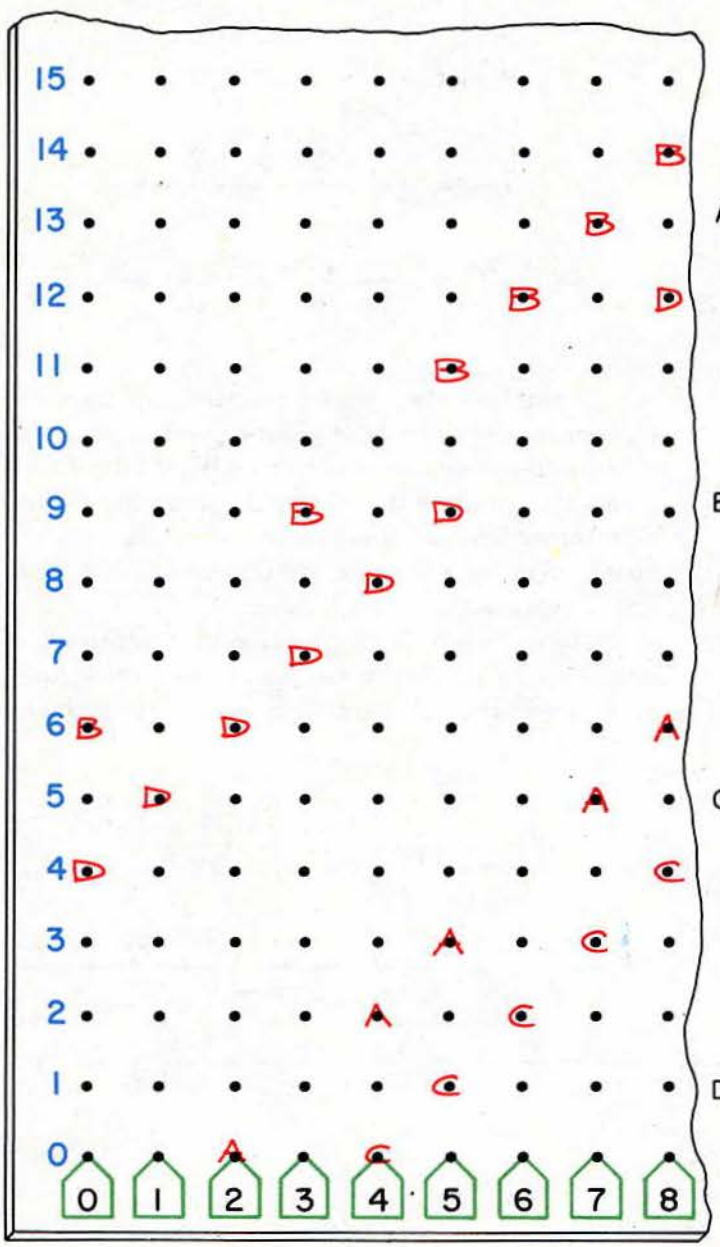
"Here's another rule. How much less than 10 is each number called out? In shorthand and words:



'Tag goes to 10 minus tag.'

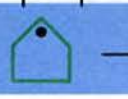
Here are some games to finish. Record the results on the scoreboard and write the rule in Bill's shorthand. Some of the numbers called out and some of Bill's answers may be so large you can't record them on the small section of the scoreboard below. In that case, you will have to imagine where the pegs would go.

(Use the game letter to show where the pegs would go for each different game.)



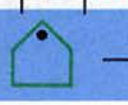
A.

Numbers called out	5	8	2	1	4	7	20
Bill's answers	3	6	0	too small	2	5	18

Bill's Rule:  - 2

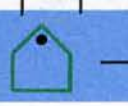
B.

Numbers called out	3	8	0	6	5	19	7
Bill's answers	9	14	6	12	11	25	13

Bill's Rule:  6 +

C.

Numbers called out	7	5	8	4	1	6	83
Bill's answers	3	1	4	0	too small	2	79

Bill's Rule:  - 4

D.

Numbers called out	2	4	8	0	3	5	1
Bill's answers	6	8	12	4	7	9	5

Bill's Rule: + 4

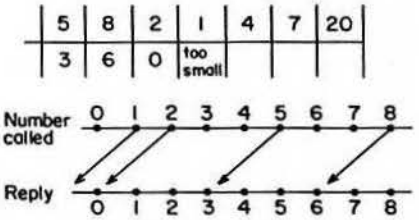
Tasks: *Pupils complete* What's My Rule? *games and rules, and record results on a grid.*

Purpose: To teach pupils to record *What's My Rule?* games in a new shorthand.

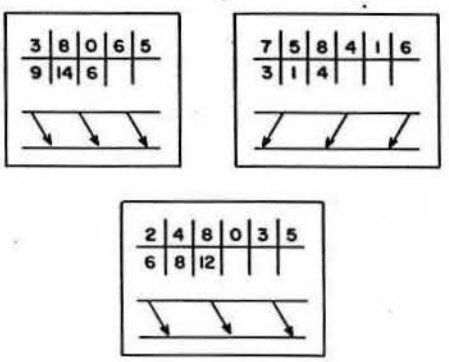
Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations; geometry.

The Lesson: It should be noticed that the points representing the answers on this pupil page form straight lines and that these lines all point in the same direction.

You might like to have pupils explore *What's My Rule?* games by representing the pairs of numbers on parallel number lines. Here is how the first example would be done:

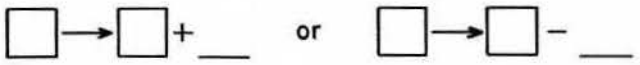


Prepare copies of such "double number lines" and have pupils diagram the other *What's My Rule?* games on this pupil page. Their diagrams should look like these:

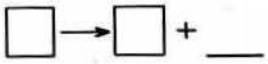


Ask pupils what they note about all their double line diagrams. They should notice that each record is composed of parallel arrows.

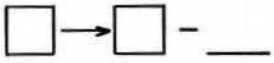
Thus far we have considered only examples of the forms:



When the rule is in the form

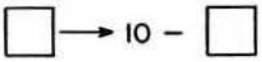


the parallel, arrows run from upper left to lower right; and when the rule is in the form

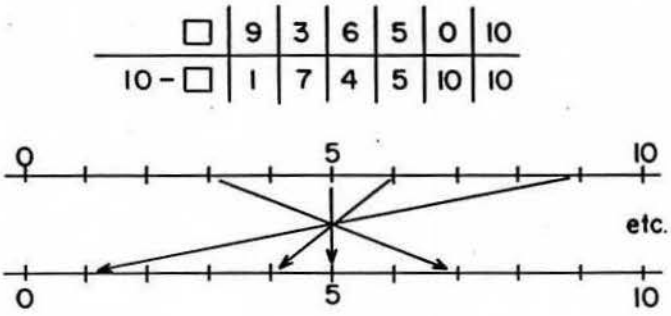


the parallel arrows run from upper right to lower left.

A surprise is in store when pupils make double line records for examples such as:



This is the second example Bill used on pupil page 123.

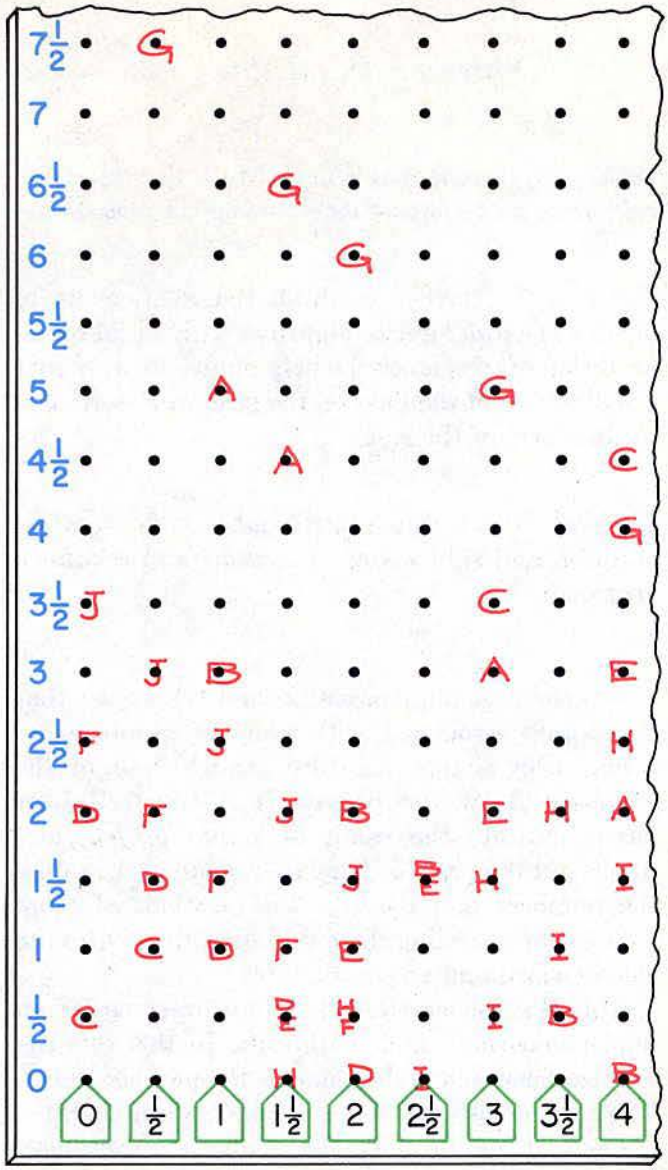


Bill changes the labels on his scoreboard.

"Thus far, we have been limited to whole numbers. I have changed the tags on the scoreboard so we can record some fractions . . .

"As we record the results on the scoreboard, watch for patterns. Sometimes the patterns will be very much like those on the previous page. Some will be different.

"I've started some games. Please discover the secret rule for each game."



A.

Bill's answers
5
2
3
0
4½

Bill's Rule: ☐ → 6 − ☐

B.

Bill's answers
0
1½
3
2
½

Bill's Rule: ☐ → 4 − ☐

C.

Bill's answers
1
3½
½
4½
16

Bill's Rule: ☐ → ½ + ☐

D.

Bill's answers
1½
½
0
2
1

Bill's Rule: ☐ → 2 − ☐

E.

Bill's answers
1
3
½
2
1½

Bill's Rule: ☐ → ☐ − 1

F.

Bill's answers
2½
2
1½
1
½

Bill's Rule: ☐ → 2½ − ☐

G.

	4	½	3	0	1½	2	7½
Bill's answers	4	7½	5	8	6½	6	½

Bill's Rule: ☐ → 8 − ☐

H.

	4	1½	3	2	3½	7	19
Bill's answers	2½	0	1½	½	2	5½	17½

Bill's Rule: ☐ → ☐ − 1½

Can you guess what patterns on the scoreboard the following two rules will lead to?

I. ☐ → ☐ − 2½

☐	2½	3	3½	4	4½*
Answers	0	½	1	1½	2

J. ☐ → 3½ − ☐

☐	0	½	1	1½	2*
Answers	3½	3	2½	2	1½

*OTHER POSSIBILITIES

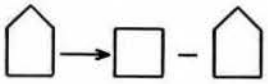
Tasks: Pupils complete What's My Rule? *games and rules, and record results on a grid.*

Purpose: To provide further practice in recording *What's My Rule?* games in shorthand. To provide for practice on a grid marked off in halves.

Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations; geometry.

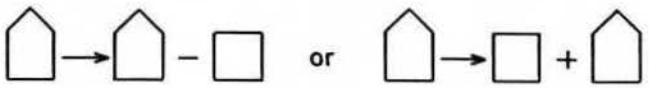
The Lesson: The answers based on this pupil page also form straight lines on the scoreboard. The lines run either from "Southwest to Northeast" or from "Northwest to Southeast." They go from SW to NE when the rule says add or subtract a certain number. These can be represented by expressions in either of these two forms:

They go from NW to SE when the number is to be subtracted from a fixed number:

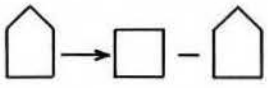


The pegboard, graph paper, or Geo-board picture helps pupils recognize the rule behind the pairing of the numbers. (An extension of this method is used in scientific work.) The pegs also help to produce accuracy. If a pupil makes a mistake, he will place a peg out of line, and he will see that something is wrong.

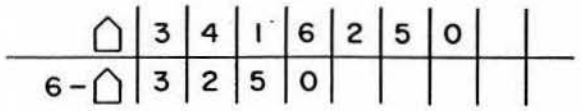
You might have pupils diagram the rules on this pupil page on double number lines. Remember that the parallel-arrows pattern results from:

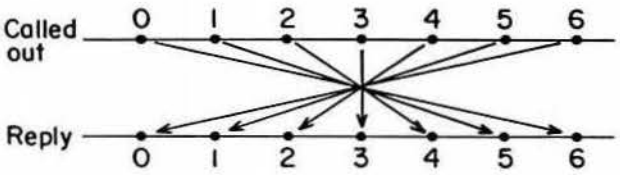


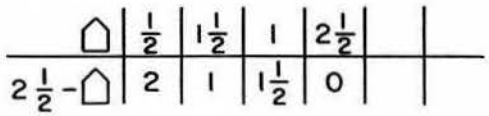
And that a different pattern results from:

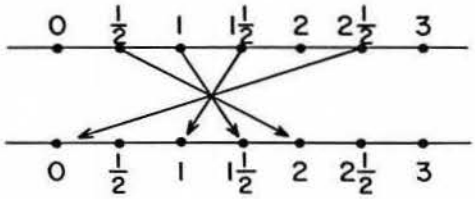


Let's look at diagrams of two games of the latter type:





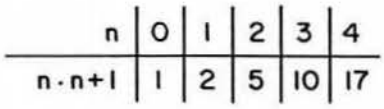


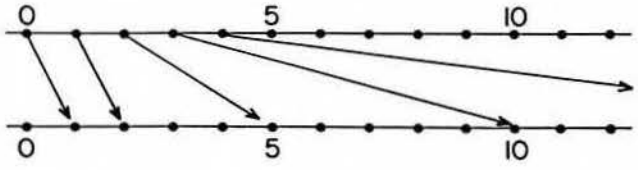


Ask pupils how they would describe this pattern. (All arrows intersect at the same point.)

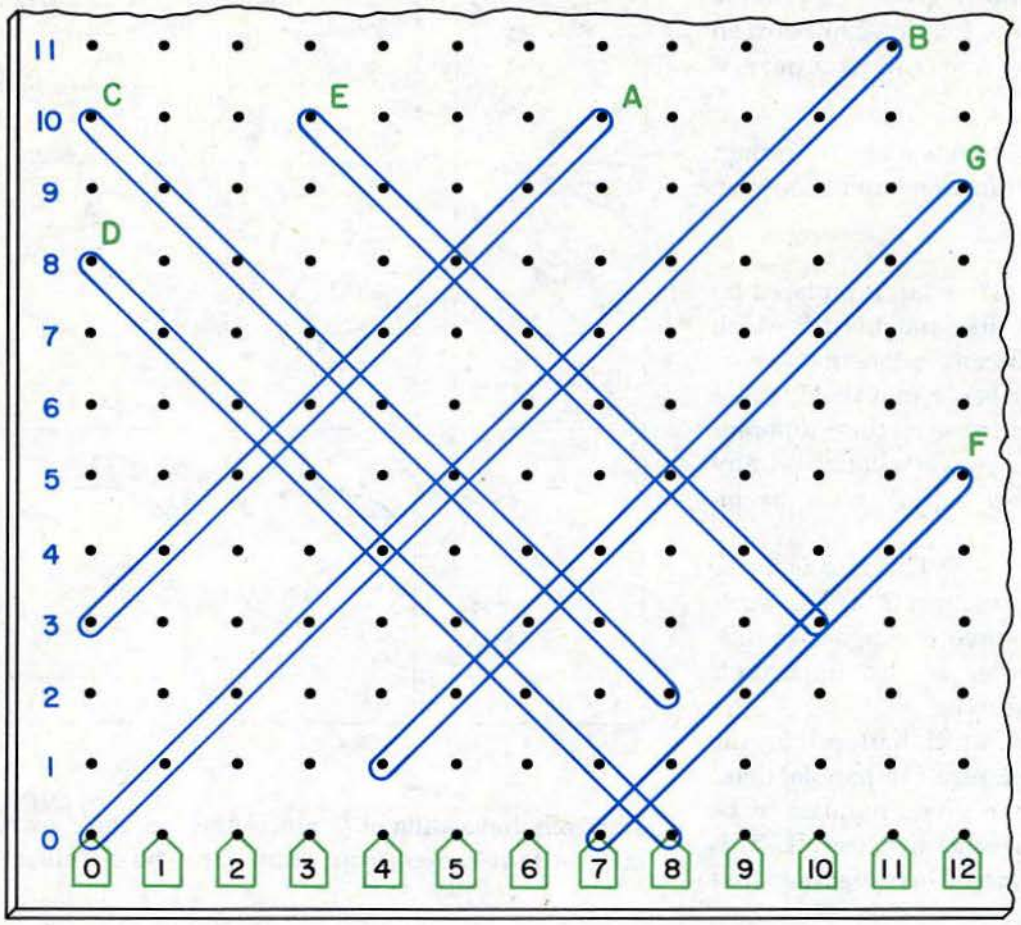
Have them make up several new *What's My Rule?* games that produce this kind of diagram. Make sure they remember how these patterns look on the pegboard or on graph paper (when the number lines are at right angles to each other).

Again we call to your attention that multiplying a number by itself does not lead to parallel arrows or to arrows through a common point. For example:





"On this scoreboard," Bill explained, "I have arranged rows of pegs to indicate results of games played. I've put a rubber band around the results of each game. Each game is lettered. I've given the rule for Game A and listed a few results. Please find all the rules used and list 7 sample results for each game."



Rule A: ⌂ → ⌂ + 3

* ⌂	⌂ + 3
7	10
5	8
0	3
1	4
2	5
3	6

Rule B: ⌂ → ⌂

* ⌂	⌂
0	0
1	1
2	2
3	3
4	4
5	5
6	6

Rule C: ⌂ → 10 − ⌂

* ⌂	10 − ⌂
8	2
7	3
6	4
5	5
4	6
3	7
2	8

Rule D: ⌂ → 8 − ⌂

* ⌂	8 − ⌂
8	0
7	1
6	2
5	3
4	4
3	5
2	6

Rule E: ⌂ → 13 − ⌂

* ⌂	13 − ⌂
10	3
9	4
8	5
7	6
6	7
5	8
4	9

Rule F: ⌂ → ⌂ − 7

* ⌂	⌂ − 7
7	0
8	1
9	2
10	3
11	4
12	5
13	6

Rule G: ⌂ → ⌂ − 3

* ⌂	⌂ − 3
4	1
5	2
6	3
7	4
8	5
9	6
10	7

* OTHER POSSIBLE ANSWERS FOR THE TABLES.

Tasks: *Pupils construct* What's My Rule? *tables and rules from the records of the games on the pegboard.*

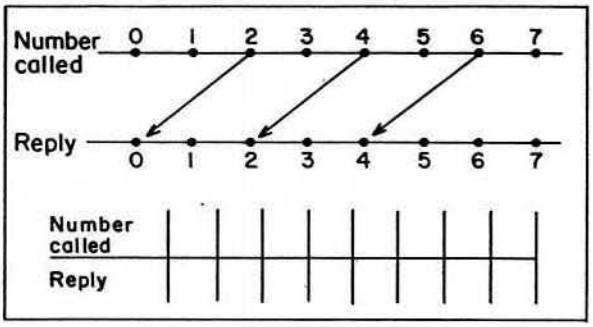
Purpose: To further establish the relationship of *What's My Rule?* games and rules with a grid representation of the games. To help pupils discover that the direction of the lines on the grid are determined by the form of the rules.

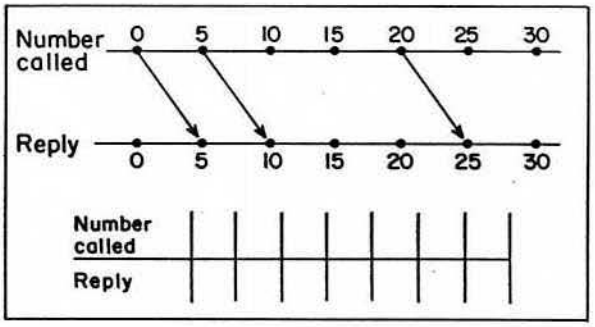
Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations; geometry.

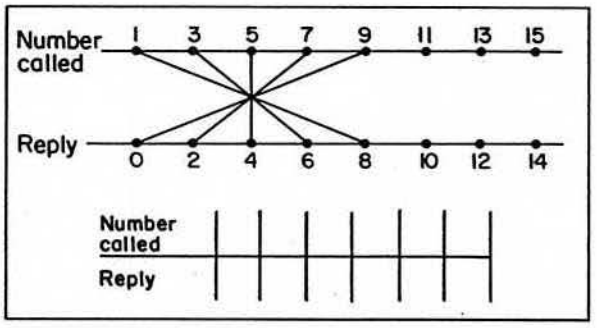
The Lesson: On pupil pages 123 and 124 we saw that the points associated with pairs of numbers determined by simple rules form straight lines on the pegboard. Now the process is reversed. Rubber bands indicate the result of games played, and pupils put the pegs in. They then write in the tables the numbers that describe the positions of these pegs. After recording these numbers, they write the rule in shorthand above the table.

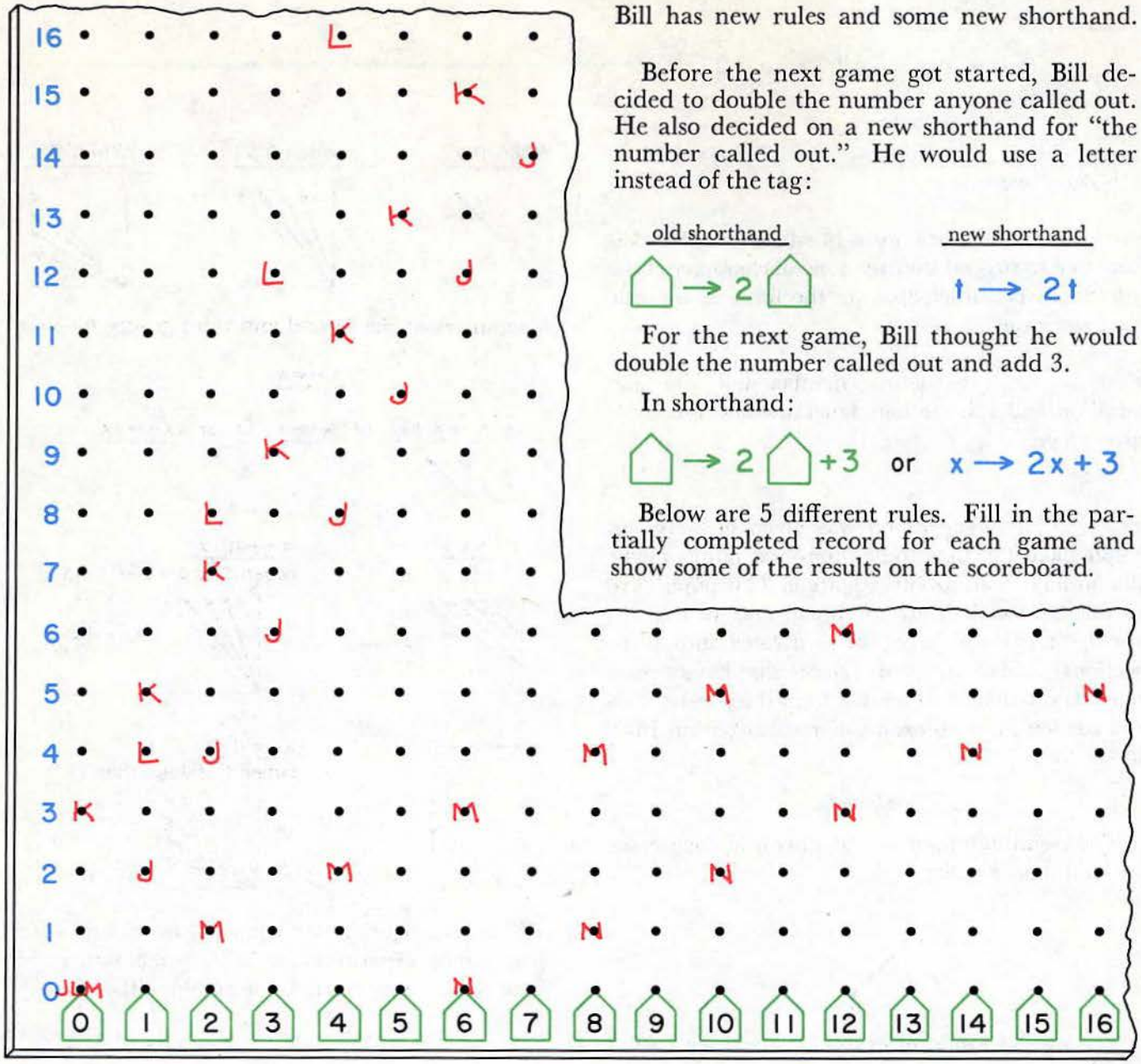
Rule B is the simplest of all, but for some reason pupils sometimes find it difficult. In this rule the number answered is the same as the number called. The number goes to itself. *Tag goes to tag* is the rule.

For each of the following double number lines, have the table completed:









Bill has new rules and some new shorthand.

Before the next game got started, Bill decided to double the number anyone called out. He also decided on a new shorthand for "the number called out." He would use a letter instead of the tag:

old shorthand	new shorthand
⌂ → 2 ⌂	t → 2t

For the next game, Bill thought he would double the number called out and add 3.

In shorthand:

⌂ → 2 ⌂ +3 or x → 2x + 3

Below are 5 different rules. Fill in the partially completed record for each game and show some of the results on the scoreboard.

Bill's Rule J

t → 2t

	4	8
*	3	6
	2	4
	1	2
	0	0
	5	10
	6	12
	7	14

Bill's Rule K

x → 2x + 3

	1	5
*	0	3
	2	7
	3	9
	4	11
	5	13
	6	15
	7	17

Bill's Rule L

s → 4s

	2	8
*	1	4
	0	0
	3	12
	4	16
	5	20
	6	24
	7	28

Bill's Rule M

r → r ÷ 2

	6	3
*	4	2
	2	1
	0	0
	8	4
	10	5
	12	6
	14	7

Bill's Rule N

q → (q ÷ 2) - 3

	8	1
*	6	0
	10	2
	12	3
	14	4
	16	5
	18	6
	20	7

* OTHER POSSIBILITIES.

Make up your own rules. Record game results on your own scoreboard.

Tasks: Pupils extend What's My Rule? *tables from rules given and then record games on the scoreboard.*

Purpose: To introduce a new shorthand using letters instead of tags for the numbers given. To provide further practice in noting the relationship between *What's My Rule?* tables and line representations of the results on a grid.

Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations; geometry.

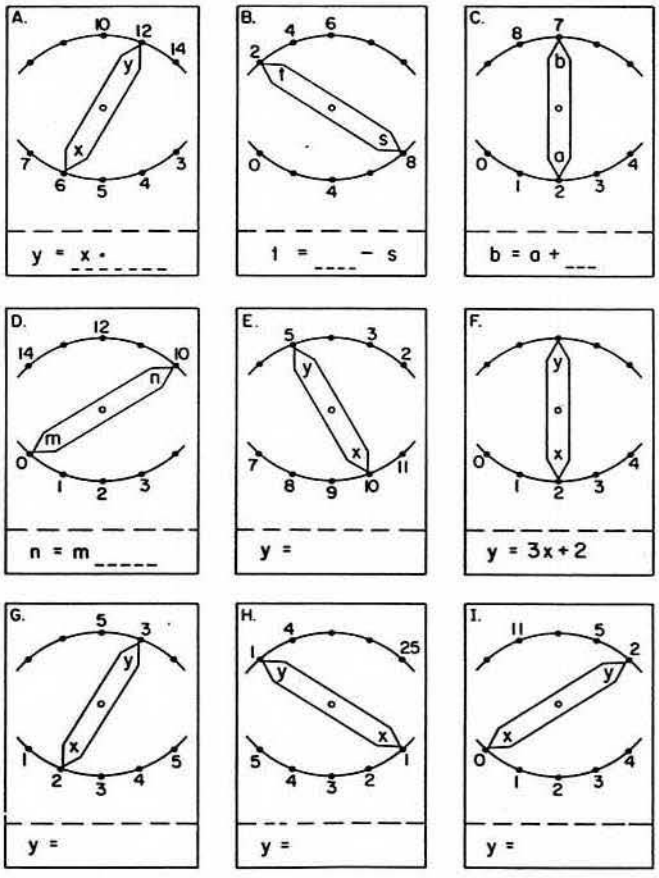
The Lesson: On pupil page 126, a tag is replaced by a letter, such as t or x. It does not matter which letter is used. So we use different letters in different exercises: t in one, x in another, r in a third, q in a fourth. There is no real need to use all these different letters. We use them so that pupils will not get any fixed idea that "the number called" must be indicated by x.

In rule N, q goes to $(q \div 2) - 3$. This means *divide the number called by 2; then subtract 3.* The order is important. Parentheses are used to emphasize this.

In $2x+3$, of course, order is also important. *Double, then add 3* is the meaning.

Notice that rules J and K, which both call for the given number to be doubled, result in parallel lines. Rules M and N call for the given number to be divided by 2, and again parallel lines result. This "coincidence" suggests a line of investigation that some interested pupil may like to follow. By taking different rules, and drawing the corresponding lines, he can observe how the direction and position of the line depends on the rule. With experience and practice, it is possible to look at a rule and immediately draw its line, or to look at a line and immediately give its rule.

Here is another way of representing rules by double number lines. After being told that they are to complete each number line and each rule, individual pupils or teams may proceed with no other preliminary discussion:



If pupils have difficulty proceeding on their own, suggest that they set up a familiar type of table:

A.

y	6	8	10	12	14
x	3	4	5	6	7

B.

t	10	8	6	4	2
s	0	2	4	6	8

It is a little more obvious now that the rule for A is $y = x \cdot 2$ or $y = 2x$ and that the rule for B is $t = 10 - s$. Other rules: C, $b = a + 5$; D, $n = m + 10$; E, $y = x - 5$; G, $y = 2x - 1$; H, $y = x \cdot x$; I, $y = 3x + 2$.

Fill in the partial record for each game and show some of the results on the scoreboard to the right of each game.

A. △ = 7 − □

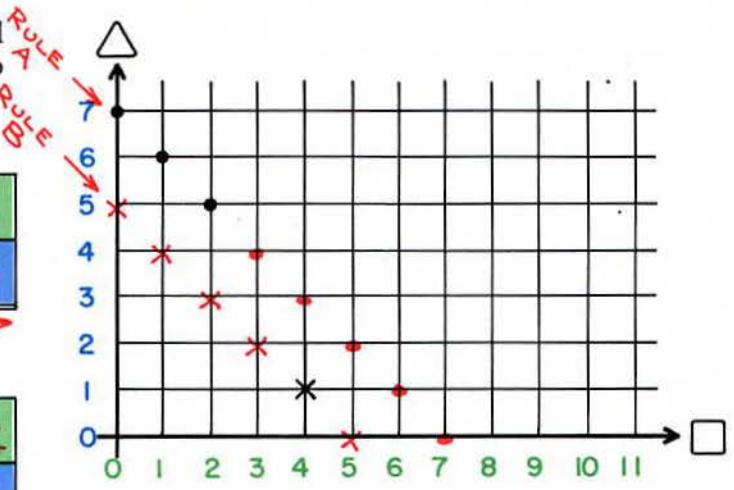
□ →	0	1	2	3	4	5	6	7
△ ↑	7	6	5	4	3	2	1	0

* →

B. △ = 5 − □

□ →	5	4	0	1	3	2	$4\frac{1}{2}$	$2\frac{1}{2}$
△ ↑	0	1	5	4	2	3	$\frac{1}{2}$	$1\frac{1}{2}$

* →



C. ◇ = ⬡

⬡ →	5	9	0	11	8	1	2	3
◇ ↑	5	9	0	11	8	1	2	3

* →

D. ◇ = ⬡ − 3

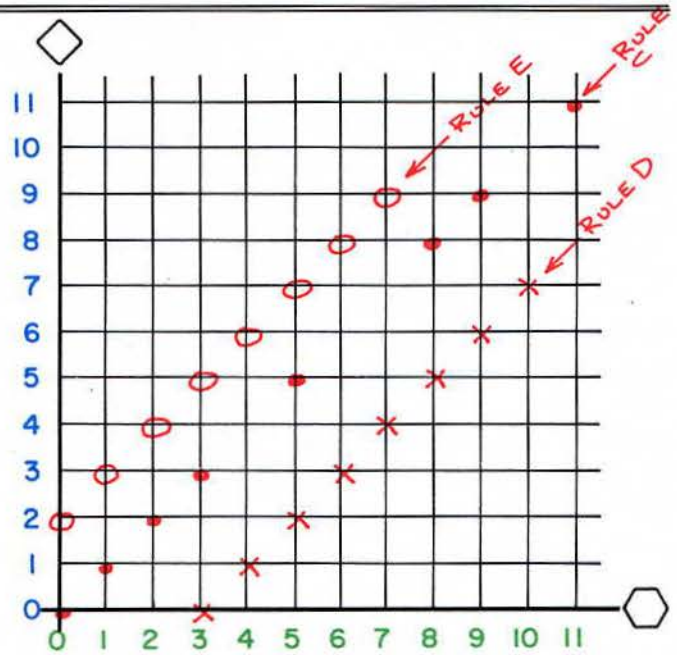
⬡	3	4	5	6	7	8	9	10
◇	0	1	2	3	4	5	6	7

* →

E. ◇ = ⬡ + 2

⬡	0	1	2	3	4	5	6	7
◇	2	3	4	5	6	7	8	9

* →



F. $y = 2 \cdot n$

n →	0	1	2	3	4	5	6	7
y ↑	0	2	4	6	8	10	12	14

* →

G. $y = 1 \cdot n$

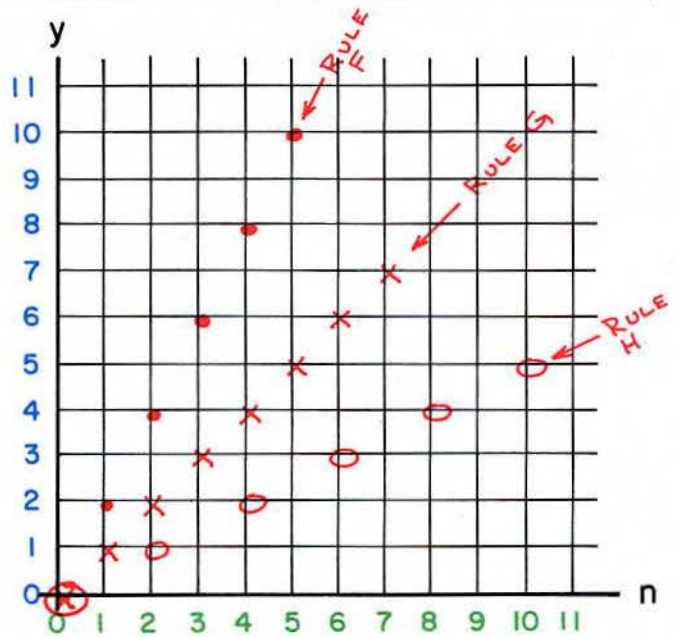
n	0	1	2	3	4	5	6	7
y	0	1	2	3	4	5	6	7

* →

H. $y = \frac{1}{2} \cdot n$

n	0	2	4	6	8	10	12	14
y	0	1	2	3	4	5	6	7

* →



* OTHER POSSIBLE ANSWERS.

Tasks: *Pupils record results of eight different* What's My Rule? *games.*

Purpose: To vary the form in which the record is kept and to suggest that the general character of the graph can be anticipated by the form of the rule that generates it.

Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations; geometry.

The Lesson: A suggestion was given in Notes on Pupil Page 126 that some interested pupils might like to follow out an investigation. This pupil page can be used to indicate the broad lines of that investigation. Pupil page 127 is divided into three sections—and each contains games that have a common aspect that is noticeable from the graphs. The top section has two examples resulting from rules in the form:

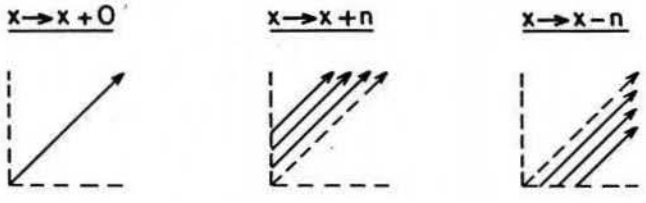
$$x \longrightarrow n - x$$

The resulting graph can be drawn in such cases without any trouble:

The second group of examples begin with rules in the form:

$$x \longrightarrow x + n \quad \text{or:} \quad x \longrightarrow x - n$$

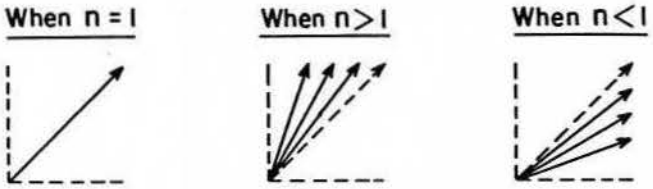
and their graphs look like one of these:



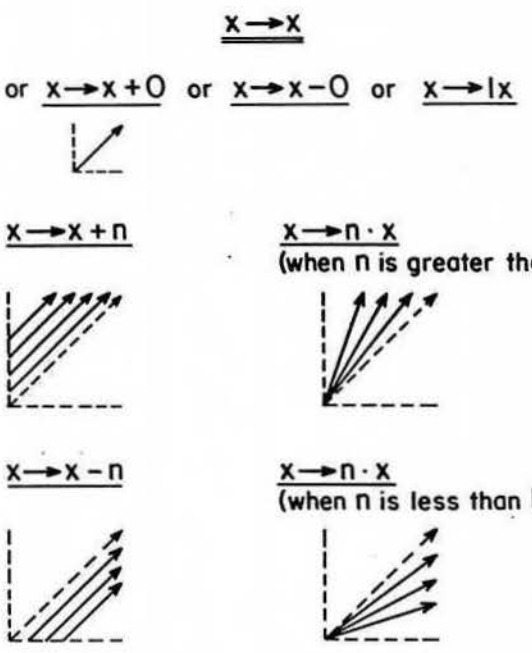
The third group brings in multiplication, which changes the direction (or slope) of the lines—making some steeper and some less steep. The rules are of the form:

$$x \longrightarrow n \cdot x$$

and their graphs look like these:

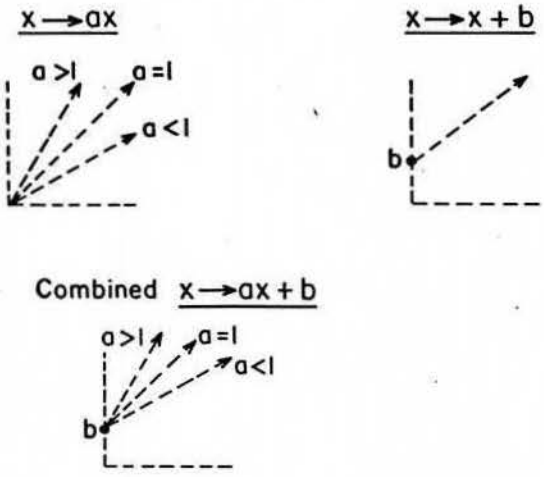


A summary of the second and third groups follows:



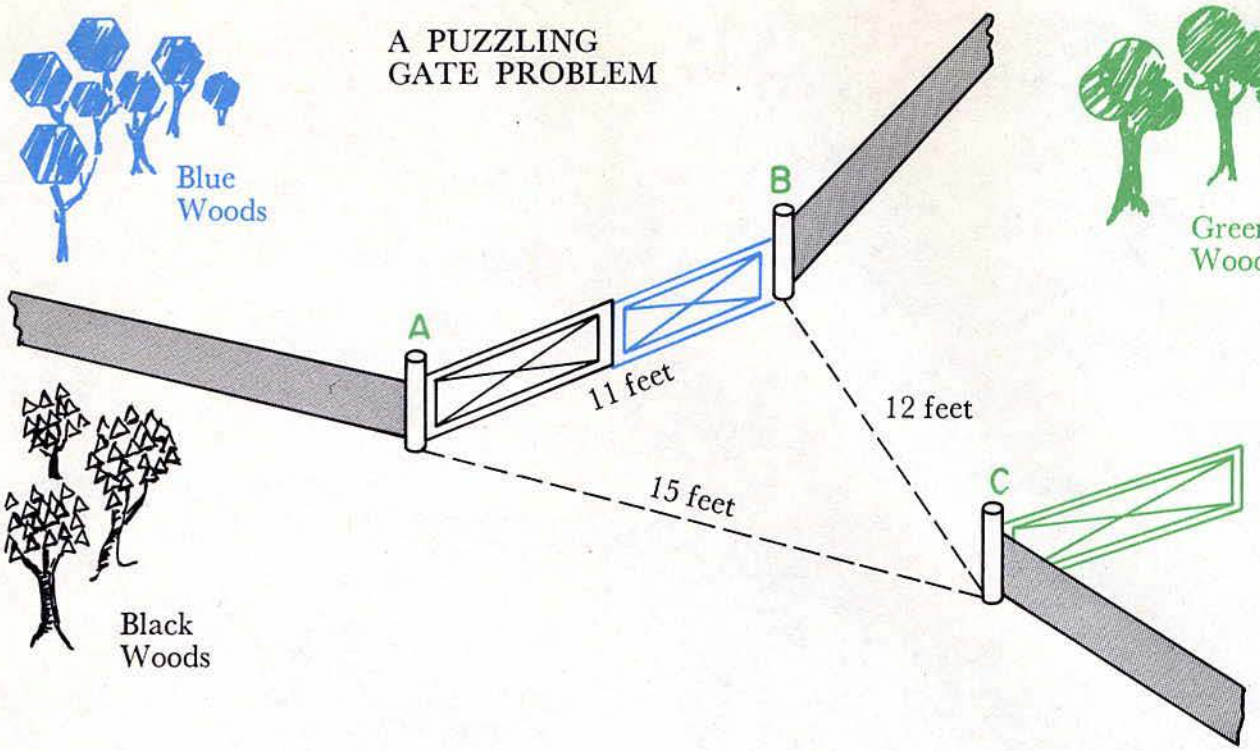
This investigation can continue for a long time with pupils experimenting with combinations of these forms. Suppose that we combine the forms:

$$x \longrightarrow ax \quad \text{and} \quad x \longrightarrow ax + b$$



Combining these basic forms anticipates an important idea in the mathematics to come.

A PUZZLING GATE PROBLEM



Farmer Jones had three gates, a black, a blue, and a green one.

There are three sections of fences on the ranch. They extend from the three posts marked A, B, and C out to the surrounding black, blue, and green woods.

There is a bear that roams through the black woods, the blue woods, and the green woods. Whenever the bear comes out of the blue woods, Farmer Jones closes the black gate and the blue gate between posts A and B (as shown above).

Whenever the bear comes out of the green woods, Farmer Jones closes the blue gate and the green gate between posts B and C.

Whenever the bear comes out of the black woods, Farmer Jones closes the black gate and the green gate between posts A and C.

In each case, Farmer Jones moves his cattle so that there is always a fence or gates in between the bear and his cattle.

The distance from post A to post C is 15 feet. The distance from post A to post B is 11 feet. The distance from post B to post C is 12 feet.

Each time the gates are closed, they meet without any gap between them, and they do not overlap.

How long are the gates?

The black gate is 7 feet long.

The blue gate is 4 feet long.

The green gate is 8 feet long.

Take your time. You may find this problem very difficult. Think about it. Try to solve it. If you don't discover the answers, don't think about the problem for a while.

Come back to the problem another day. Try to solve it again.

NOTE: If you solve the problem, please don't tell others. Each person should discover the answer for himself!

12345 J 70 69 68 67 66

Tasks: Pupils study the gate problem and try to determine the lengths of the black, blue, and green gates.

Purpose: To provide pupils with a problem which they can solve by trial and error using strategies learned in this book.

Unifying Ideas: Structure; addition and subtraction; functions and relations; geometry; measurement.

The Lesson: This puzzle may amuse the students. It can be solved by simple algebra. But the pupils do not yet have the knowledge necessary to do this. They will probably solve it by trial and error.

Once the size of gate A has been fixed, the sizes of gates B and C are easy to find. For example, if gate A were 10 feet long, gate B would have to be 1 foot (to fill the 11 feet between posts A and B), and gate C would have to be 5 feet (to fill the 15 feet between posts A and C). But the 1-foot gate on post B and the 5-foot gate on post C would leave a big gap in the middle of the 12-foot span from B to C. So 10 feet is not the length of gate A. A pupil may next try some other size for this gate, and see whether it makes things better or worse. After a few trials, he should discover the correct solution.

Copy these problems as a suitable finale to Level D. Pupils or teams may wish to devise similar reconstructions for others to solve.

	1	2	3	4
s + t	16	35	64	42
r	6	2	13	0
t − s	8	5	14	28
t ÷ r	2	10	3	35
s	4	15	25	7
r + t	18	22	52	35
rs	24	30	325	0
s + r	10	17	38	7
st + r	54	302	975	245
t	12	20	39	35

	5	6	7	8
b+10	12			
a+c+2				17
ab−3		24		
b+c−1			12	8
a+7	17			
a÷b+4		7	6	
ac+5				
c−3	1			
(a+b)2			18	20
bc−7		21		

9

x , y	f
3 , 9	54
5 , 8	
, 7	84
, 8	144
9 ,	108

f = 2xy

10

a , b	n
8 , 4	
25 , 5	
42 , 7	
, 13	9
, 9	16

n = 3a ÷ b

11

c , r	t
2 , 7	
7 , 4	
9 , 17	
, 9	96
, 3	90

t = c(r+3)

12

m , x	d
5 , 3	
9 , 5	
, 4	12
25 ,	23
, 13	7

d = 2m−3x

APPENDIX

THE EXPANDING LANGUAGE OF MATHEMATICS

Many unfamiliar expressions are heard in current discussions of mathematics. Some teachers are interested; others are disturbed; some are confused. The purpose of this appendix is to illustrate the extremely simple ideas indicated by some of the new words and expressions.

Because the concepts dealt with are not at all new, many teachers will prefer to discuss them in the language they have always used. They can do this without any qualms of conscience. All teachers are urged to use new words and expressions *only after* they have a firm comprehension of the ideas, have familiarized themselves with the language, and can use it comfortably. The ideas are important: the language is useful only if it helps present the ideas more clearly.

1. Sets

Mathematicians often borrow words from the general language and give them a specific mathematical meaning. They usually prefer this to inventing words of their own. Such a word is *set*. We are familiar with such expressions as:

"*Set* it on the table."
"The sun will *set* at 5:30 P.M."
"That stage *set* is ingenious."
"I'm going to *set* my hair."
"Let's play a *set* of tennis."
"The cement will *set* by morning."

None of these has much, if anything, to do with the meaning mathematicians have given the word *set*. Their meaning has more in common with:

"How many pieces in the *set* of dishes?"
"Breakfast *sets* for $24.99."
"He has a *set* of ivory chessmen."
"I got a gold-plated pen and pencil *set*."

They looked at the word *set* as it might be used in place of *group*, *collection*, *combination*, *aggregate*, *family*, etc., and then proceeded to give that general idea a precise mathematical meaning.

Sets in Mathematics. Many uses of the word *set* in mathematics have a rather familiar ring. Here are a few in which the word might be replaced by *group* or *collection* as those words are used in everyday discourse.

The set of all counting numbers between 10 and 15 has 11, 12, 13, and 14 as its members.

Mathematicians use braces to suggest a set, so they might indicate this particular set of numbers by:

$$\{11, 12, 13, 14\}$$

A mathematician might wish to direct your attention to all the counting numbers and zero that are multiples of 7, in which case he would write:

$$\{0, 7, 14, 21, 28, \ldots\}$$

The three dots are a notation for "and so on in a like manner."

The members of a set need not be numbers. We might wish to consider, for example:

The set of all cities in the United States with populations in excess of one million.

The set of all objects in my purse.

The set of redheaded boys.

The set of all triangles.

The set of all letters in the alphabet that are never used as vowels.

2. Union of Sets

One way of combining sets is called "union," for which the symbol $\cup$ is most commonly used.

(1) $\{A, B\} \cup \{A, C, E\} = \{A, B, C, E\}$
What letters occur either in the set with members A and B or in the set with members A, C, and E? Answer: the letters A, B, C, and E.

(2) $\{1, 5, 4\} \cup \{4, 6, 1\} = \{1, 4, 5, 6\}$ (or $\{4, 6, 5, 1\}$, etc.)

(3) $\{Al, Ed\} \cup \{Carl, Ed\} = \{Al, Ed, Carl\}$

(4) $\{m, n\} \cup \{r, s, t\} = \{m, n, r, s, t\}$

(5) $\{7, 2, 9\} \cup \{9, 2\} = \{7, 2, 9\}$

(6) $\{a, b, c\} \cup \{c, b, a\} = \{a, b, c\}$

(7) $\{b, x, y\} \cup \{x\} = \{\_\_, \_\_, \_\_\}$

(8) $\{Jane\} \cup \{Jane, Mary\} = \{\_\_, \_\_\}$

(9) $\{P, Q\} \cup \{R, S, T\} = \{\_\_, \_\_, \_\_, \_\_, \_\_\}$

(10) $\{(3, 4), (7, 9)\} \cup \{(7, 9), (4, 3)\} = \{(3, 4), (7, 9), (4, 3)\}$

(11) $\{(7, 0)\} \cup \{(0, 7)\} = \{(7, 0), (0, 7)\}$

(12) $\{(7, 0)\} \cup \{(7, 0)\} = \{(7, 0)\}$

(13) $\{(1, 8)\} \cup \{(8, 1)\} = \{(\ ,\ ), (\ ,\ )\}$

(14) $\{(0, 0), (0, 1), (1, 1)\} \cup \{(1, 0), (0, 1)\} = \{(\ ,\ ), (\ ,\ ), (\ ,\ ), (\ ,\ )\}$

(15) $\{(a, b), (b, a)\} \cup \{(b, b), (a, b)\} = \{\_\_\_\_\_\_\_\_\_\_\}$

(16) {r, s, t} ∪ {s, t, a, r} = {_____}
(17) {17, 23, 47} ∪ {32, 47, 71} = {___, ___, ___, ___, ___}
(18) {(1, 1), (2, 2)} ∪ {(2, 1), (1, 1)} = {_____}

3. Intersection of Sets

Another way of combining sets is called "intersection," for which the symbol ∩ is most commonly used.

(1) {R, S, T} ∩ {T, O} = {T}
What letters occur both in the set with members R, S, and T and in the set with members T and O? Answer: Only the letter T.
(2) {a, b, c, d} ∩ {c, a, e} = {a, c}
(3) {5, 7, 3} ∩ {57, 73, 7} = {7}
(4) {a, b, d} ∩ {b, e, a, d} = {a, b, d} (or {b, a, d}, etc.)
(5) {3, 6, 9, 12, 18} ∩ {4, 8, 12, 16} = {12}
(6) {(0, 0), (1, 5), (2, 10)} ∩ {(5, 1), (1, 5)} = {(1, 5)}
(7) {Bill, Sam, Tom} ∩ {Tom, Sam, Ed} = {Sam, Tom}
(8) {Bill, Dick, Tom} ∩ {Carl, Dick} = {_____}
(9) {(7, 1), (5, 3), (0, 8)} ∩ {(1, 7), (0, 8)} = {(,)}
(10) {8, 4, 9, 1} ∩ {4, 1, 9, 8} = {_____}
(11) {(28, 3), (23, 8)} ∩ {(23, 8), (28, 3)} = {_____}
(12) {(0, 1), (1, 0), (1, 2)} ∩ {(2, 1), (1, 0), (0, 1)} = {_____}
(13) {0, 2, 4, 6, 8, 10, 12} ∩ {0, 3, 6, 9, 12, 15} = {_____}
(14) {1, 3, 5, 7, 9, 11} ∩ {2, 4, 6, 8, 10, 12} = {_____}
In example 14 there are no members common to both sets, so the intersection in this case is called "the empty set."

4. The Empty Set

While this expression sounds a bit strange at first, the idea is very simple, as you can see by referring to example 14 above. The symbol used to indicate the empty set is ∅. So we might say: the empty set = { } = ∅.

(1) (A, R, T) ∩ (B, C) = { } = ∅
Which letters occur both in the set with members A, R, and T and in the set with members B and C? Answer: none. So we say that the intersection is the empty set.
(2) {5, 7} ∩ {0, 6, 12} = ∅
(3) {a, b} ∪ {c} = {a, b, c}
(4) {a, b} ∩ {c} = ∅
(5) {3, 7, 11} ∩ { } = ∅
Which numbers are both in the set with members 3, 7, and 11 and the empty set? Answer: none. Since the symbol { }—braces with nothing listed between them—might be confusing, the empty

set is usually indicated by the symbol $\varnothing$. So $\{\quad\} \cap \{\quad\} = \{\quad\}$ would be written: $\varnothing \cap \varnothing = \varnothing$.

(6) $\varnothing \cap \{1, 2, 3\} =$

(7) $\varnothing \cup \{1, 2, 3\} = \{1, 2, 3\}$

(8) $\{(0, 8), (9, 1)\} \cap \{(9, 8)\} =$

(9) $\{\text{Alice, Mary}\} \cap \{\text{Bill}\} =$

(10) $\{4, 8, 12\} \cup \{0, 5, 10\} = \{\_\_, \_\_, \_\_, \_\_, \_\_, \_\_\}$

(11) $\{4, 8, 12\} \cap \{0, 5, 10\} =$

5. Number of a Set

We are often concerned with the number of members in a set. Since we are referring to numbers, we can add and subtract. We write a capital N before the name of the set to indicate that we are concerned only with the number of members in the set.

(1) $N\ \{1, 4, 17\} = 3$

(2) $N\ \{a, b, x, y, z\} = 5$

(3) $N\ \{a, b, c\} + N\ \{x, y, z\} = 3 + 3 = 6$

(4) $N\ \{a, b, c\} + N\ \{a, b, c\} = 6$

(5) $\{a, b, c\} \cup \{a, b, c\} = \{a, b, c\}$

(6) $N\ \{4, 6, 8, 9\} + N\ \{6, 8, 11\} = 7$

(7) $\{4, 6, 8, 9\} \cup \{6, 8, 11\} = \{4, 6, 8, 9, 11\}$

(8) $\{4, 6, 8, 9\} \cap \{6, 8, 11\} = \{6, 8\}$

(9) $N\ \{x, y, z, a\} - N\ \{1, 2, 4\} = 1$

(10) $N\ \{0, 1, 2, 3\} + N\ \{0\} = 5$

(11) $N\ \{\quad\} = N\ \varnothing = 0$

6. Solution Sets for Mathematical Sentences

In the following examples let us agree to limit ourselves to the counting numbers and zero. Thus, for the open sentence:

$$\triangle + \square = 2$$

there are only three pairs of numbers that could be used to convert it into a true sentence:

$$(0, 2) \ldots\ldots 0 + 2 = 2$$
$$(1, 1) \ldots\ldots 1 + 1 = 2$$
$$(2, 0) \ldots\ldots 2 + 0 = 2$$

The *solution set* of this open sentence is the set whose members are (0, 2), (1, 1), and (2, 0).

In this case, where pairs of numbers are used, we must agree that the first member of the pair is associated with the first placeholder in the mathematical sentence, and that the second member is associated with the second placeholder. Such a pair is called an *ordered pair*. Since addition is commu-

tative, this seems rather trivial, but for the operations that are not commutative this idea becomes very important.

Here are additional examples of solution sets.

Examples	Open Sentence	Solution Set
(1)	$\triangle + \square = 2$	$\{(0,2),(1,1),(2,0)\}$
(2)	$\square \times 1 = \square$	$\{0, 1, 2, 3, 4, \ldots\}$

because $\underline{0} \times 1 = \underline{0}$ $\quad 3 \times 1 = 3$

$\underline{1} \times 1 = \underline{1}$ $\quad 4 \times 1 = 4$

$2 \times 1 = 2$ etc.

(3)	$\triangle - \triangle = 0$	$\{0, 1, 2, 3, 4, \ldots\}$
(4)	$\square + \square = \square \times \square$	$\{0, 2\}$

because $0 + 0 = 0 \times 0$

$2 + 2 = 2 \times 2$

There are just two numbers in the solution set for example 4 because 0 and 2 are the only numbers whose doubles are the same as their squares.

(5)	$\square - \triangle = 6$	$\{(6,0),(7,1),(8,2),\ldots\}$

because $6 - 0 = 6$

$7 - 1 = 6$

$8 - 2 = 6$ etc.

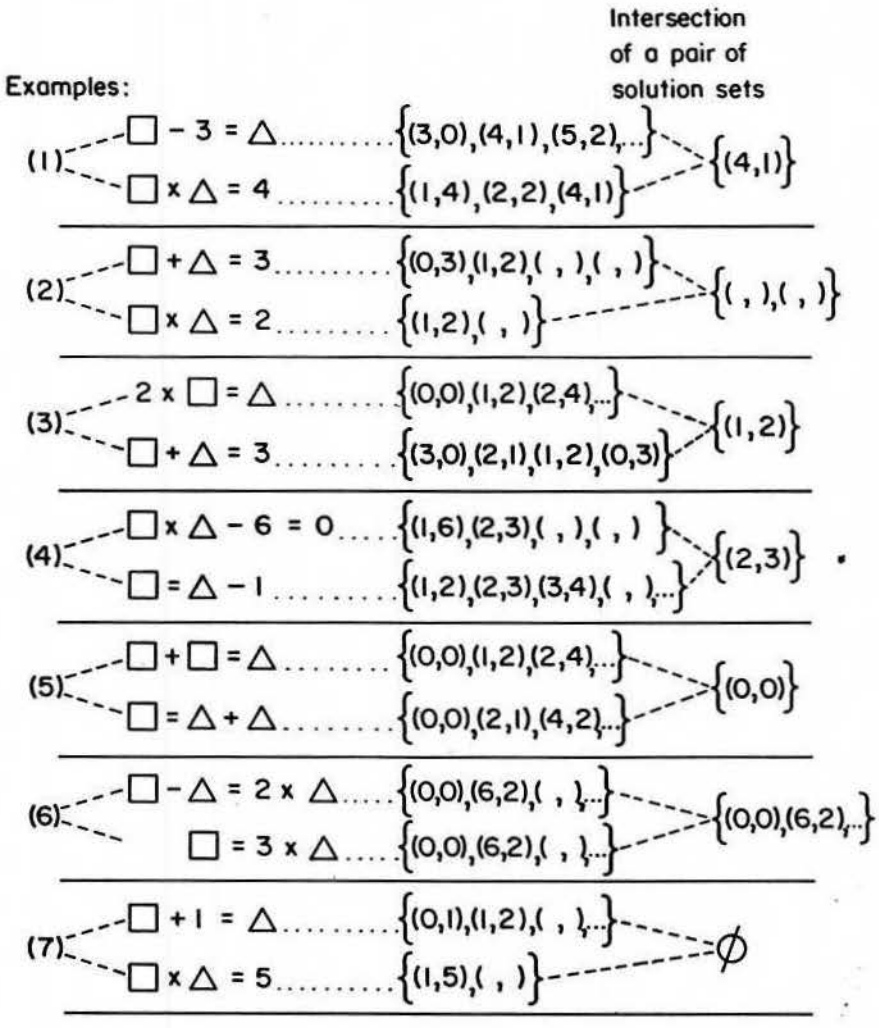
(6)	$\square - 1 = \square$	ϕ

In example 6, the solution set is the empty set because there is no number which is 1 less than itself:

$0 - 1 = 0$ is a false statement
$1 - 1 = 1$ is a false statement
$2 - 1 = 2$ is a false statement
$3 - 1 = 3$ is a false statement
etc.

(7)	$\square + \triangle = \triangle + \square$	All pairs of numbers
(8)	$\square + 9 = 24$	$\{15\}$
(9)	$10 - \triangle = 7$	$\{\quad\}$
(10)	$3 \times \square - 1 = 14$	$\{\quad\}$
(11)	$\triangle \times \square = 6$	$\{(\ ,\ ),(\ ,\ ),(\ ,\ ),(\ ,\ )\}$

7. Intersection of Solution Sets



8. Subsets of Sets

Examples:

Consider Set A = {M, N}. There are four subsets of Set A. They are {M}, {N}, {M, N}, and ∅.

Consider Set B = {Carl}. There are two subsets of Set B. They are {Carl} and ∅.

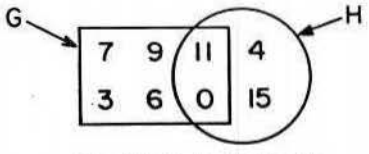
Consider Set C = {1, 2, 3}. There are eight subsets of Set C. They are {1}, {2}, {3}, {1, 2} {1, 3}, {2, 3}, {1, 2, 3}, and ∅.

Were you tempted to say that {N, M} is a subset of Set A? Or that {3, 2, 1} is a subset of Set C? Here order does not concern us, for {M, N} and {N, M} are the same set.

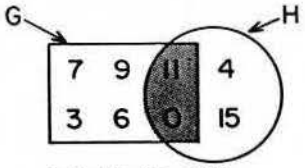
Notice that (a) every set is a subset of itself
and
(b) the empty set is a subset of every set.

9. AND and OR

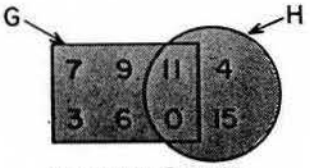
Many of our examples depended on the distinction between the meanings of *and* and *or* as mathematicians use these words. We can explore these ideas in diagrams. Suppose that we are talking about G, the set of numbers written in the rectangle, and H, the set of numbers written in the circle.



G = (7,9,11,3,6,0)
H = (11,4,0,15)

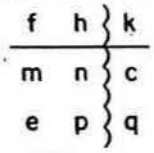


in both G and H
G ∩ H = (11, 0)

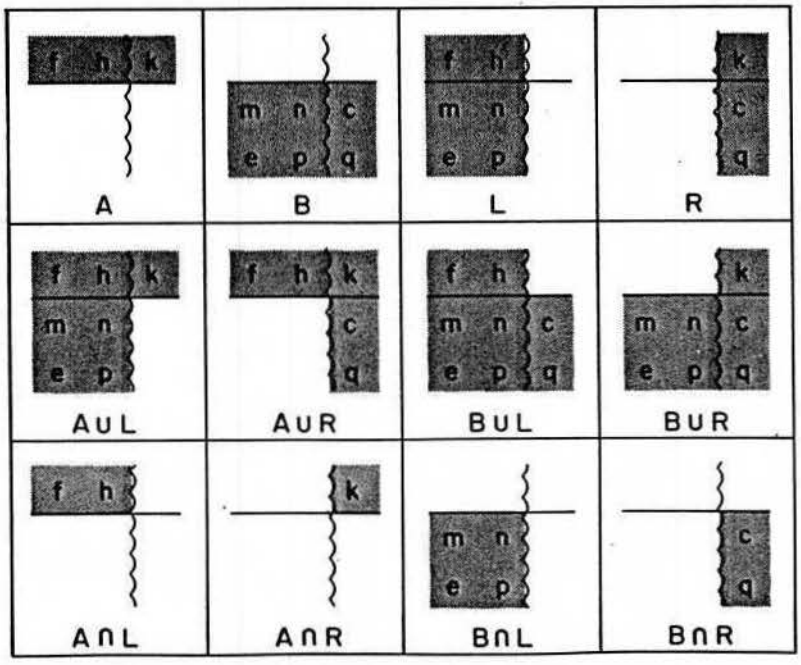


in either G or H
G ∪ H = (7,9,11,4,3,6,0,15)

Let us consider another diagram, involving nine letters.



A = the set of letters above the horizontal line = {f, h, k}
B = the set of letters below the horizontal line = {m, n, c, e, p, q,}
L = the set of letters to the left of the wavy line = {f, h, m, n, e, p}
R = the set of letters to the right of the wavy line = {k, c, q}



$A \cup L = \{f, h, k, m, n, e, p\}$
$A \cap L = \{f, h\}$
$A \cup R = \{f, h, k, c, q\}$
$A \cap R = \{k\}$
$B \cup L = \{\_\_\_\_\_\}$
$B \cap L = \{\_\_\_\_\_\}$
$B \cup R = \{\_\_\_\_\_\}$
$B \cap R = \{\_\_\_\_\}$
$A \cup B = \{f, h, k, m, n, c, e, p, q\}$
$A \cap B = \varnothing$
$L \cup R = \{\_\_\_\_\_\}$
$L \cap R =$

This interpretation of *and* and *or* is a mathematical interpretation, but in many ways this usage conforms with conversational English. The idea is disarmingly simple. Once children grasp the concepts indicated by these two words, we have a way to ask many questions about a single example such as the diagram above.